1 Introduction

In this paper we introduce the idea of induced representations and use them help find character tables. An induced representation is a representation which is constructed from a representation of a subgroup. We will show how to construct the induced representation and how to construct the character directly from the character of the subgroup (thus avoiding matrices involved with the representations). We will then use induced representations to help find the character table for the Quaternions, A_4, and S_4.

2 Induced Representations

Definition 2.1. *Suppose that $H \leq G$ and ρ is a degree m representation of H (over either $\mathbb{R}$ or $\mathbb{C}$). Then,*

$$\dot{\rho}(g) = \begin{cases} \rho(g) & \text{if } g \in H \\ 0_m & \text{if } g \notin H \end{cases}$$

Let $n = G : H$ (the number of cosets of H in G). Consider a representative from each coset $\{g_1 = 1, g_2, g_3, \ldots, g_n\}$.

Definition 2.2. *The induced representation $\rho^G(g)$ is constructed from matrices of $\dot{\rho}$. The (i,j) subblock of the induced representation is $\dot{\rho}(g_i^{-1} g g_j)$.*

$$\rho^G(g) = \begin{pmatrix} \dot{\rho}(g_1^{-1} g g_1) & \cdots & \dot{\rho}(g_1^{-1} g g_j) & \cdots & \dot{\rho}(g_1^{-1} g g_n) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \dot{\rho}(g_i^{-1} g g_1) & \cdots & \dot{\rho}(g_i^{-1} g g_j) & \cdots & \dot{\rho}(g_i^{-1} g g_n) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \dot{\rho}(g_n^{-1} g g_1) & \cdots & \dot{\rho}(g_n^{-1} g g_j) & \cdots & \dot{\rho}(g_n^{-1} g g_n) \end{pmatrix}$$

Recalling that n is the number of cosets and the degree of ρ is m,

Theorem 2.3. *$\rho^G(g) : G \to GL(V)$ is a representation where $deg(V) = mn$.*

Proof. The proof that $\rho^G(g)$ is a representation is rather cumbersome and does not add much value to the readers understanding. It is left as an exercise for Rolf.

Regarding the degree of $\rho^G(g)$, we have n $m \times m$ matrices in each row and in each column. Thus the resulting matrix for $\rho^G(g)$ is a $mn \times mn$, so $\rho^G(g)$ is degree mn. □

To help demonstrate how to construct the induced representation, we present the following examples.

Example 2.4. *We will construct the representation ρ^{D_4} of D_4 induced by the representation ρ of H, where*

$$D_4 = \langle r, c \mid r^4 = c^2 = 1, cr = r^{-1}c \rangle$$

$$H = \langle r \rangle \qquad D_4 = H \cup cH$$

and

$$\rho : H \to GL(\mathbb{C})$$

$$\rho(r) = [i]$$

Our coset representatives g_1 and g_2 are 1 and c, respectively.

It is useful to calculate a table of values for $\dot{\rho}$ (given by defintion 2.1).

$$\dot{\rho}(1) = [1] \qquad \dot{\rho}(r) = [i]$$

$$\dot{\rho}(r^2) = [-1] \qquad \dot{\rho}(r^3) = [-i]$$

$$\dot{\rho}(c) = \dot{\rho}(rc) = \dot{\rho}(r^2c) = \dot{\rho}(r^3c) = [0]$$

Then, using definition 2.2, we find

$$\rho^{D_4}(r) = \begin{pmatrix} \dot{\rho}(1^{-1}r1) & \dot{\rho}(1^{-1}rc) \\ \dot{\rho}(c^{-1}r1) & \dot{\rho}(c^{-1}rc) \end{pmatrix} = \begin{pmatrix} \dot{\rho}(r) & \dot{\rho}(rc) \\ \dot{\rho}(r^3c) & \dot{\rho}(r^3) \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Similarly for $c \in D_4$, we have

$$\rho^{D_4}(c) = \begin{pmatrix} \dot{\rho}(1^{-1}c1) & \dot{\rho}(1^{-1}cc) \\ \dot{\rho}(c^{-1}c1) & \dot{\rho}(c^{-1}cc) \end{pmatrix} = \begin{pmatrix} \dot{\rho}(c) & \dot{\rho}(1) \\ \dot{\rho}(1) & \dot{\rho}(c) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Example 2.5. *As a second example, we will construct another induced representation of D_4 using a slightly more difficult representation of H. We will let*

$$\mu : H \to GL(\mathbb{R}^2)$$

$$\mu(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and define $\dot{\mu}$ according to definition 2.1. Since μ is a degree 2 representation of H, and $|D_4 : H| = 2$, we expect to see 4×4 matrices. Sure enough, appealing to definition 2.2 we find

$$\mu^{D_4}(r) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and

$$\mu^{D_4}(c) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

3 Induced Character

Now that we have a feel for finding induced representations, it would be nice if we could find their characters without finding the representations (as these can easily turn into large matrices).

Let ρ be a representation of H and let χ be the character of ρ. Consider the trace of $\dot{\rho}$ (see definition 2.1).

$$tr(\dot{\rho}(g)) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

This motivates the following definition:

Definition 3.1. $\dot{\chi}(g) = \begin{cases} \chi(g) & \textit{if } g \in H \\ 0 & \textit{if } g \notin H \end{cases}$

Definition 3.2. $\chi^G(g) = \sum_{i=1}^{n} \dot{\chi}(g_i^{-1} g g_i)$

Theorem 3.3. *The character of the induced representation $\rho^G(g)$ is $\chi^G(g)$*

Proof. We wish to find the character of the induced representation $\rho^G(g)$. Note that each submatrix along the diagonal has the form $\dot{\rho}(g_i^{-1} g g_i)$. Now consider the trace of the induced representation.

$$\begin{aligned} tr(\rho^G(g)) &= \sum_{i=1}^{n} tr(\dot{\rho}(g_i^{-1} g g_i)) \\ &= \sum_{i=1} \dot{\chi}(g_i^{-1} g g_i) \text{ by definition 3.1} \\ &= \chi^G(g) \end{aligned}$$

□

Note that with this definition induced character involves a sum over representatives from each coset of G. This could cause a problem: what if different choices of representatives result in a different induced character? Fortunately this does not happen, and we will show an equivalent defintion in the next section which demonstrates this.

4 Another Definition of Induced Character

Definition 4.1. $\chi^G(g) = \dfrac{1}{|H|} \sum_{x \in G} \dot{\chi}(x^{-1} g x)$

We will show definition 3.2 is equivalent to definition 4.1

Let G be a group, $g \in G$ with $h \in H \leq G$ and χ a character of H. By definition, we have that $h^{-1}gh$ is a conjugate of g. We also have a useful result from abstract algebra that says

$$h^{-1} g h \in H \iff g \in H \tag{1}$$

First, we will show a useful lemma.

Lemma 4.2. *$\dot{\chi}(g) = \dot{\chi}(h^{-1}gh)$, where $\dot{\chi}$ is defined as*

$$\dot{\chi}(g) = \begin{cases} \chi(g) & g \in H \\ 0 & g \notin H \end{cases}$$

Proof. We will consider two cases. In the first case, we have $g \in H$. From (1) we know that $h^{-1}gh$ must also be in H. Then it follows from the Definition 3.2 that $\dot{\chi}(g) = \chi(g)$ and $\dot{\chi}(h^{-1}gh) = \chi(h^{-1}gh)$. Since χ is a class function (recall that all characters are class functions) and $h^{-1}gh$ and g are conjugates, we have that $\chi(g) = \chi(h^{-1}gh)$. Stringing these equalities together gives us our result

$$\dot{\chi}(g) = \chi(g) = \chi(h^{-1}gh) = \dot{\chi}(h^{-1}gh)$$

In the second case, we have $g \notin H$, and our result follows immediately from the definition of $\dot{\chi}$:

$$\dot{\chi}(g) = \dot{\chi}(h^{-1}gh) = 0$$

This is sufficient to show that $\dot{\chi}(g) = \dot{\chi}(h^{-1}gh)$. □

We will now continue our proof that definitions 3.2 and 4.1 are equivalent.

Let g_iH be some coset of H in G. We will refer to g_i as the *coset representative* of g_iH. Consider the quantity

$$\dot{\chi}\big((g_i h)^{-1} g (g_i h)\big)$$

Rewriting $(g_ih)^{-1}$ as $h^{-1}g_i^{-1}$ and taking advantage of associativity, we have

$$= \dot{\chi}\big(h^{-1}(g_i^{-1} g g_i) h\big)$$

From the closure of the group G, we have $g_i^{-1}gg_i \in G$, and by Lemma 4.2

$$= \dot{\chi}(g_i^{-1}gg_i)$$

So far, we have

$$\dot{\chi}\big((g_ih)^{-1}g(g_i)h\big) = \dot{\chi}(g_i^{-1}gg_i)$$

For notational purposes, we will let $x = g_ih \in g_iH$. Then,

$$\dot{\chi}(x^{-1}gx) = \dot{\chi}(g_i^{-1}gg_i) \tag{2}$$

Now, for each coset, $g_iH \subset G$, we can take the sum of (2) over all of the elements in g_iH

$$\sum_{x \in g_iH} \dot{\chi}(x^{-1}gx) = \sum_{x \in g_iH} \dot{\chi}(g_i^{-1}gg_i) \tag{3}$$

But notice that the summand on the right hand side of (3) has no x dependence, and so when we evaluate the sum, we just get $|g_iH|$ copies of $\dot{\chi}(g_i^{-1}gg_i)$

$$\sum_{x \in g_iH} \dot{\chi}(x^{-1}gx) = |g_iH|\dot{\chi}(g_i^{-1}gg_i)$$

noting that $|g_iH| = |H|$ (all cosets are created equal) and rearanging terms gives

$$\dot{\chi}(g_i^{-1}gg_i) = \frac{1}{|H|}\sum_{x \in g_iH} \dot{\chi}(x^{-1}gx) \tag{4}$$

Plugging this into definiton 3.2, we find

$$\begin{aligned}
\chi^G(g) &= \sum_{i=1}^{n} \dot{\chi}(g_i^{-1}gg_i) \\
&= \sum_{i=1}^{n} \left(\frac{1}{|H|} \sum_{x \in g_iH} \dot{\chi}(x^{-1}gx) \right) \\
&= \frac{1}{|H|} \sum_{i=1}^{n} \sum_{x \in g_iH} \dot{\chi}(x^{-1}gx) \\
&= \frac{1}{|H|} \sum_{g \in G} \dot{\chi}(x^{-1}gx)
\end{aligned}$$

where in our last step we have noticed that summing over all elements in all cosets is equivent to summing over all elements of our group (this last statement holds because the cosets of g_iH partition G). This establishes that definitions 3.2 and 4.1 for the induced character are equivalent.

5 Normal Subgroups

Theorem 5.1. *Let $H \trianglelefteq G$ and let χ be a character of G. Then $\chi^G(g) = 0$ for all $g \notin G$.*

Proof. Since H is a normal subgroup of G, we have $gHg^{-1} = H$ for all $g \in G$. This means that, for any $g \in G$, conjugation by g moves no element of G that is inside H out of H, and, more importantly, moves no element of G that is outside H into H. Then, for $g \notin G$, all of our $\dot{\chi}(x^{-1}gx)$ in the summation in Definition 4.1 are equal to zero (by Definition 3.1). □

6 Character Table for the Quaternion Group

We are now interested in studying the character table of Quaternions. Recall that the group of Quaternions are

$$\mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$$

where $i^2 = j^2 = k^2 = ijk = -1$. Note that this group is non-Abelian.

There are five conjugacy classes: $\{1\}$, $\{-1\}$, $\{\pm i\}$, $\{\pm j\}$, and $\{\pm k\}$. Since each element's inverse is its own conjugacy class, by homework problem 36a, we know the entries in the character table are real.

Note that $|\mathbb{H}| = 8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$ is the only way to decompose 8 as the sum of 5 integer squares. Thus this gives us the first column of the character table. As usual, we have the trivial representation as our first row. Thus,

$\mathbb{H}$	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
χ_1	1	1	1	1	1
χ_2	1				
χ_3	1				
χ_4	1				
χ_5	2				

Note that the Quaternions have three subgroups of index 2, namely $\{\pm 1, \pm i\}$, $\{\pm 1, \pm j\}$, and $\{\pm 1, \pm k\}$. We know from homework problem 15b (and the fact that these are degree 1) that the character for the elements in the group are $+1$ and the character for the elements not in the group are -1. This gives us the next three rows of the character table.

$\mathbb{H}$	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2				

To help us find the last row of the character table, lets look at the character induced by the subgroup $N = \{1, -1\}$. Let $\rho : N \to GL(\mathbb{R})$ be the representation $\rho(1) = 1$ and $\rho(-1) = -1$ (with character χ).

Recall that $N \trianglelefteq G$ if and only if $gNg^{-1} = N$ for all $g \in G$. Note that

$$\begin{aligned} gNg^{-1} &= \{g1g^{-1}, g(-1)g^{-1}\} \\ &= \{1gg^{-1}, (-1)gg^{-1}\} \text{ (since 1 and -1 are in the center of } G) \\ &= \{1, -1\} \\ &= N \end{aligned}$$

Thus $N \trianglelefteq G$. This means that we can use Theorem 5.1 to save time in computations. Lets now find $\chi^G(g)$ when $g \in N$. Note that $n = |\mathbb{H}|/|N| = 8/2 = 4$. The induced character (from Definition 3.2) is

$$\begin{aligned}
\chi^G(1) &= \sum_{i=1}^{4} \dot{\chi}(g_i^{-1} 1 g_i) = \sum_{i=1}^{4} \dot{\chi}(1) \\
&= \sum_{i=1}^{4} \chi(1) = \sum_{i=1}^{4} 1 \\
&= 4 \\
\chi^G(-1) &= \sum_{i=1}^{4} \dot{\chi}(g_i^{-1}(-1) g_i) = \sum_{i=1}^{4} \dot{\chi}(-1) \\
&= \sum_{i=1}^{4} \chi(-1) = \sum_{i=1}^{4} -1 \\
&= -4
\end{aligned}$$

Coupling this with Theorem 5.1, we find the induced character is

$\mathbb{H}$	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
χ^G	4	-4	0	0	0

Notice that the inner product of χ^G with χ_1, χ_2, χ_3, and χ_4 is zero. χ^G is a linear combination of irreducible characters, but cannot contain χ_1, χ_2, χ_3 or χ_4. This implies that $\chi^G = k * \chi_5$. Looking at the character table thus far, we see that we must have $k = 2$. This completes the character table for the Quaternions.

$\mathbb{H}$	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

7 Character Table for A_4

We are now interested in finding the character table for A_4 (which is the subgroup of even permutations in S_4). First note that $|A_4| = |S_4|/2 = 4!/2 = 12$. There are four conjugacy classes, namely (1), (123), (132), and (12)(34). Thus, since $|A_4| = 12 = 1^2 + 1^2 + 1^2 + 3^2$ is the only way to write 12 as the sum of 4 integer squares, these are the degreese of the irreducible representations. Thus, the character table thus far looks like

A_4	(1)	(123)	(132)	(12)(34)
χ_1	1	1	1	1
χ_2	1			
χ_3	1			
χ_4	3			

Now notice that (123) has order 3 in S_4. Thus $\rho((123)^3)) = \rho((1)) = 1 = \rho((123))^3$ (where ρ is the representation corresponding to χ_2). Since ρ is degree 1, $\rho((123))$ must be a third root of unity. Since we already have the trivial representation, it must be either ω or ω^2 (where $\omega = (-1 + \sqrt{3})/2$). Thus its character is either ω or ω^2. Note that (132) has order 3 in S_4. By the same logic above, we find that the character of (132) is either ω or ω^2.

Since we have two remaining degree 1 representations, we find for $\chi_2((123)) = \omega$, $\chi_2((132)) = \omega^2$, $\chi_3((123)) = \omega^2$, and $\chi_3((132)) = \omega$. To satisfy orthonormality, we find that $\chi_2((12)(34)) = \chi_3((12)(34)) = 1$.

A_4	(1)	(123)	(132)	(12)(34)
χ_1	1	1	1	1
χ_2	1	ω	ω^2	1
χ_3	1	ω^2	ω	1
χ_4	3			

At this point we can use algebra to find the last row. Suppose that the last row looks like

A_4	(1)	(123)	(132)	(12)(34)
χ_4	3	a	c	b

If this were the character, it would be orthogonal to the first three rows. This implies that

A_4	(1)	(123)	(132)	(12)(34)
χ_4	3	c	a	b

would also be a character. Since we only have one row left to fill, this cannot be the case. Thus $a = c$, and the last row is of the form

A_4	(1)	(123)	(132)	(12)(34)
χ_4	3	a	a	b

Since the irreducible characters form an orthonormal basis, we find that

$$\langle \chi_4, \chi_4 \rangle = \frac{1}{12}\left(9 + 8a^2 + 3b^2\right) = 1$$

$$\langle \chi_4, \chi_1 \rangle = 3 + 8a + 3b = 0$$

When we find solutions to these equations, we find that either $a = 0$ and $b = -1$ or $a = -6/11$ and $b = -5/11$. One more inner-product will find the unique solution.

$$\begin{aligned}\langle \chi_2, \chi_4 \rangle &= 3 + 4a\omega + 4a\omega^2 + 3b \\ &= 3 + 4a(\omega + \omega^2) + 3b \\ &= 3 - 4a + 3b = 0\end{aligned}$$

This shows that the solution is $a = 0$ and $b = -1$. Thus we find the character table for A_4 is

A_4	(1)	(123)	(132)	(12)(34)
χ_1	1	1	1	1
χ_2	1	ω	ω^2	1
χ_3	1	ω^2	ω	1
χ_4	3	0	0	-1

8 Character Table for S_4

Finally, we will calculate the character table of S_4. The order of S_4 is 24, and there are five conjugacy classes: (1), (12), (123), (1234), and (12)(34). Since the only way to write 24 as a sum of five squares is $1^2 + 1^2 + 2^2 + 3^2 + 3^2 = 24$, we know that the orders of the irreducible representations of S_4 are 1, 1, 2, 3, and 3. At this point we can begin filling part of the character table of S_4

S_4	(1)	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_2	1				
χ_3	2				
χ_4	3				
χ_5	3				

where χ_1 is the trivial representation. Earlier in the semester on a homework problem, we showed that in any permutation group there is an irreducible degree one representation that sends every even element to 1 and every odd element to -1. Calling this representation χ_2, we can add it to our table

S_4	(1)	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2				
χ_4	3				
χ_5	3				

In the hopes of finding another irreducible representation of S_4, we will decompose $\rho : S_4 \to GL(\mathbb{R}^4)$, the natural representation of S_4 in $\mathbb{R}^4$ (which we know to be reducible), into a direct sum of irreducible representations, in accordance with Maschke's Theorem. Recall that

$$\rho\big((12)\big) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho\big((124)\big) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho\big((1432)\big) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho\big((12)(34)\big) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Since all of $\rho(g)$, $g \in S_4$, are permutation matrices, it is clear that the vector $\langle 1,1,1,1\rangle$ is unchanged by any $\rho(g)$. This motivates a change of basis, orchestrated by the matrices

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 0 & -1 & -1 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 1/4 & 1/4 & -1/4 & -1/4 \end{pmatrix}$$

In our new basis, our four matrices (one from each non-trivial conjugacy class) become

$$A^{-1}\rho\big((12)\big)A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A^{-1}\rho\big((124)\big)A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/2 & -1/2 & -1 \\ 0 & 1/2 & 1/2 & -1 \\ 0 & 1/2 & -1/2 & 0 \end{pmatrix}$$

$$A^{-1}\rho\big((12)(34)\big)A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A^{-1}\rho\big((1432)\big)A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/2 & -1/2 & 1 \\ 0 & -1/2 & -1/2 & -1 \\ 0 & -1/2 & 1/2 & 0 \end{pmatrix}$$

and it is immediately clear that ρ can be decomposed into the direct sum of a degree three representation and a degree one representation; we are primarily interested in the degree three representation. Reading down the diagonals of our four (one from each conjugacy class) 3×3 submatrices, we can quickly calculate the character, χ_4, of our brand new degree three representation of S_4, taking care to remember the trivial conjugacy class.

	(1)	(12)	(123)	(1234)	(12)(34)
χ_4	3	1	0	-1	-1

Including χ_4 in our character table for S_4, we now have

S_4	(1)	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2				
χ_4	3	1	0	-1	-1
χ_5	3				

Recall from the χ_4 of the character table for A_4 that

	(1)	(123)	(132)	(12)(34)
χ	3	0	0	-1

To aid us in our endeavors, we will construct a character of S_4 induced from χ. Recall that S_4 has five conjugacy classes: (1), (12), (123), (1234), (12)(34), and that A_4 is a normal subgroup of S_4 with $|S_4 : A_4| = 2$. Our two coset representatives will be (1) and (12). From Definition 3.2, we have

$$\begin{aligned}
\chi^{S_4}\big((1)\big) &= \dot{\chi}\big((1)^{-1}(1)(1)\big) + \dot{\chi}\big((12)^{-1}(1)(12)\big) = 6 \\
\chi^{S_4}\big((12)\big) &= \dot{\chi}\big((1)^{-1}(12)(1)\big) + \dot{\chi}\big((12)^{-1}(12)(12)\big) = 0 \\
\chi^{S_4}\big((123)\big) &= \dot{\chi}\big((1)^{-1}(123)(1)\big) + \dot{\chi}\big((12)^{-1}(123)(12)\big) = 0 \\
\chi^{S_4}\big((1234)\big) &= \dot{\chi}\big((1)^{-1}(1234)(1)\big) + \dot{\chi}\big((12)^{-1}(1234)(12)\big) = 0 \\
\chi^{S_4}\big((12)(34)\big) &= \dot{\chi}\big((1)^{-1}(12)(34)(1)\big) + \dot{\chi}\big((12)^{-1}(12)(34)(12)\big) = -2
\end{aligned}$$

And so our new character of S_4 induced from χ is

	(1)	(12)	(123)	(1234)	(12)(34)
χ^{S_4}	6	0	0	0	-2

It is clear that χ^{S_4} is not irreducible, but it may still prove quite helpful. Taking the innner product of χ^{S_4} with χ_1 and χ_2, we find

$$\langle \chi^{S_4}, \chi_1 \rangle = \langle \chi^{S_4}, \chi_2 \rangle = 0$$

which means that there are no copies of χ_1 or χ_2 in χ^{S_4}. However, taking the inner product of χ^{S_4} with χ_4 reveals

$$\langle \chi^{S_4}, \chi_4 \rangle = 1$$

And so there *is* a copy of χ_4 in χ^{S_4} which we can immediately subtract off, giving

$$\chi^{S_4} - \chi_4 = \mu$$

where μ takes on the values

	(1)	(12)	(123)	(1234)	(12)(34)
μ	3	-1	0	1	-1

Noting that $\langle \mu, \mu \rangle = 1$, we conclude that μ is irreducible, and insert it into our character table of S_4 as χ_5

S_4	(1)	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2				
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	1	-1

We can also construct another character of S_4, this time induced from a representation χ' of $S_3 < S_4$. Recall that S_3 has three conjugacy classes, (1), (12), and (123), and that A_3 has four cosets in S_4.

$$S_4 = (1)S_3 \cup (14)S_3 \cup (24)S_3 \cup (34)S_3$$

Our four coset representatives will be (1), (14), (24), and (34). We will use a character χ' from S_3 that we solved for earlier in the semester

	(1)	(12)	(123)
χ'	2	0	-1

From Definition 3.2, we have

$$\begin{aligned}
\chi'^{S_4}\big((1)\big) &= \dot{\chi}\big((1)^{-1}(1)(1)\big) + \dot{\chi}\big((12)^{-1}(1)(12)\big) + \dot{\chi}\big((13)^{-1}(1)(13)\big) + \\
&\quad + \dot{\chi}\big((14)^{-1}(1)(14)\big) \\
&= 8 \\
\chi'^{S_4}\big((12)\big) &= \dot{\chi}\big((1)^{-1}(12)(1)\big) + \dot{\chi}\big((12)^{-1}(12)(12)\big) + \\
&\quad + \dot{\chi}\big((13)^{-1}(12)(13)\big) + \dot{\chi}\big((14)^{-1}(12)(14)\big) \\
&= 0 \\
\chi'^{S_4}\big((123)\big) &= \dot{\chi}\big((1)^{-1}(123)(1)\big) + \dot{\chi}\big((12)^{-1}(123)(12)\big) + \\
&\quad + \dot{\chi}\big((13)^{-1}(123)(13)\big) + \dot{\chi}\big((14)^{-1}(123)(14)\big) \\
&= -1 \\
\chi'^{S_4}\big((1234)\big) &= \dot{\chi}\big((1)^{-1}(1234)(1)\big) + \dot{\chi}\big((12)^{-1}(1234)(12)\big) + \\
&\quad + \dot{\chi}\big((13)^{-1}(1234)(13)\big) + \dot{\chi}\big((14)^{-1}(1234)(14)\big) \\
&= 0 \\
\chi'^{S_4}\big((12)(34)\big) &= \dot{\chi}\big((1)^{-1}(12)(34)(1)\big) + \dot{\chi}\big((12)^{-1}(12)(34)(12)\big) + \\
&\quad + \dot{\chi}\big((13)^{-1}(12)(34)(13)\big) + \dot{\chi}\big((14)^{-1}(12)(34)(14)\big) \\
&= 0
\end{aligned}$$

And so our new character of S_4 induced from χ' is

	(1)	(12)	(123)	(1234)	(12)(34)
χ'^{S_4}	8	0	-1	0	0

Taking the inner product of χ'^{S_4} with each of the irreducible characters that we have already solved for gives

$$\langle \chi'^{S_4}, \chi_1 \rangle = \langle \chi'^{S_4}, \chi_2 \rangle = 0$$
$$\langle \chi'^{S_4}, \chi_4 \rangle = \langle \chi'^{S_4}, \chi_5 \rangle = 1$$

Which motivates the formation of μ' by subtracting a single copy of χ_4 and of χ_5 from χ'^{S_4}

$$\mu' = \chi'^{S_4} - \chi_4 - \chi_5$$

where μ' takes on the values

	(1)	(12)	(123)	(1234)	(12)(34)
μ'	2	0	-1	0	2

Since $\langle \mu', \mu' \rangle = 1$, it is immediately clear that μ' is irreducible, and we set $\chi_3 = \mu$ and complete our table for S_4.

S_4	(1)	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	1	-1