Surgeries on Finite-Dimentional Tight Frames

B. Backup C. LeBailly

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Abstract

In this paper we examine the conditions under which it is possible to do tight frame surgeries. A surgery involves replacing some vectors with others to form a tight frame. We show that it is always possible to add unit vectors to a collection of unit vectors to form a unit tight frame. It is also possible to do a surgery on a unit tight frame, so as the number of unit vectors removed is less than or equal to the number of unit vectors replaced. We also find conditions which are necessary for performing tight frame surgeries which retain the original frame bound, and that these conditions are sufficient for the case in which one vector is replaced by two.

1 Background and Notation

1.1 Linear Algebra

Recall that a *Hilbert space* is a complete vector space with an associated inner product. Also recall that $\{e_i\}_{i=1}^n \subset \mathcal{H}$ is an *orthonormal basis* if and only if $\{e_i\}_{i=1}^n$ is a basis and

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \text{ for all } 1 \leq i, j \leq n$$

We also use topics covered in an advanced Linear Algebra course. Given a linear operator T from a Hilbert space $\mathcal{H}$ to $\mathcal{K}$, there exists a unique linear operator T^*, the *adjoint* of T, from $\mathcal{K}$ to $\mathcal{H}$ such that for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$,

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

If we have $T : \mathcal{H} \mapsto \mathcal{H}$, a linear operator, then T is *positive* if it is self-adjoint ($T^* = T$) and $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$.

Orthonormal basis are studied and used widely since they have many useful properties. Two such properties are:

Proposition 1.1. *Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Given $x \in \mathcal{H}$, the unique coefficents for the linear expansion of x are given by*

$$x = \sum_{i=1}^{n} \langle x, e_i \rangle \, e_i \quad \forall \, x \in \mathcal{H}$$

This then yeilds another useful equation:

Proposition 1.2. *(Parseval's Identity). Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Then, for every $x \in \mathcal{H}$,*

$$\|x\|^2 = \sum_{i=1}^{n} | \langle x, e_i \rangle |^2$$

(Note: The proofs of these propositions can be found in several texts, such as [1]).

These properties make orthonormal bases very useful. However, requiring that the vectors of a basis are orthogonal and normal restricts the number of possible orthonormal bases for a particular Hilbert space. Are there other spanning sets that satisfy these properties? If not, are there spanning sets that satisfy similar properties? These questions motivate the study of frames.

1.2 Frames

In a given Hilbert space $\mathcal{H}$, a set of vectors $\{x_i\}_{i=1}^k \subset \mathcal{H}$ is a frame for $\mathcal{H}$ if there exist positive constants A and B (the lower and upper frame bounds for $\{x_i\}_{i=1}^k$, respectively) such that, for every $x \in \mathcal{H}$,

$$A\|x\|^2 \leq \sum_{i=1}^{k} | \langle x, x_i \rangle |^2 \leq B\|x\|^2$$

It should be noted that, in a finite-dimensional Hilbert space, a frame is equivalent to a spanning set. There exist, however, some special types of frames. The rest of this paper is motivated by the study of tight frames, which are frames where the lower and upper frame bounds are equal; i.e., for some $A > 0$,

$$A\|x\|^2 = \sum_{i=1}^{k} | \langle x, x_i \rangle |^2$$

There are also some special types of tight frames. A *Parseval frame* is a tight frame with frame bound $A = 1$. Note that a Parseval frame, by definition, satisfies Parseval's Identity (Propsition 1.2). Another type of tight frame is a *unit tight frame*. This is a tight frame where all the frame vectors are unit vectors.

Example Let $\mathcal{H} = \mathbb{R}^2$ and let $\{x_1, x_2, x_3\}$ be the vectors

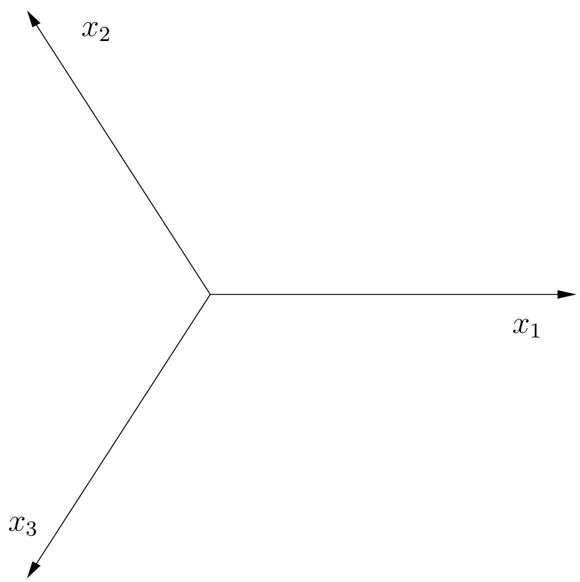

Figure 1: The Mercedes-Benz frame

$$\left\{ \sqrt{\frac{2}{3}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ \sqrt{\frac{3}{2}} \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ -\sqrt{\frac{3}{2}} \end{bmatrix} \right\}$$

This collection of vectors is a Parseval frame, and often called the Mercedes-Benz frame. We can verify that this collection of vectors is a Parseval frame by showing it satisfies $\|x\|^2 = \sum_{i=1}^{k} |\langle x, x_i \rangle|^2$. Also, we can verify by direct computation that

$$x = \langle x, x_1 \rangle x_1 + \langle x, x_2 \rangle + \langle x, x_3 \rangle x_3$$

This is called the reconstruction formula, similar to Proposition 1.2 for orthonormal bases. We will latter show this holds for all Parseval frames. To do so, we need some more tools.

1.2.1 The Frame Operator

To help us study frames, we introduce the following operators: the analysis operator, the synthesis operator, and the frame operator.

Definition Let $\{x_i\}_{i=1}^k \subset \mathcal{H}$. The analysis operator of $\{x_i\}_{i=1}^k$, $\Theta : \mathcal{H} \mapsto \mathbb{C}^k$ defined by

$$\Theta x = \begin{bmatrix} \langle x, x_1 \rangle \\ \vdots \\ \langle x, x_k \rangle \end{bmatrix} = \sum_{i=1}^{k} \langle x, x_i \rangle e_i$$

where $\{e_i\}_{i=1}^k$ is the standard orthonormal basis for $\mathbb{C}^k$.

The analysis operator takes a vector in $\mathcal{H}$ and maps it to its coordinates in $\mathbb{C}^k$, with respect to the frame $\{x_i\}_{i=1}^k$.

Definition The adjoint of the analysis operator (Θ^*) is called the *synthesis operator*.

Let $\{e_i\}_{i=1}^k$ be the standard orthonormal basis for $\mathcal{H}$ and let $\{x_i\}_{i=1}^k$ be a frame in $\mathcal{H}$. For each $y \in \mathcal{H}$ and each j, we find

$$\begin{aligned}\langle y, \Theta^* e_j \rangle &= \langle \Theta y, e_j \rangle \text{ by the definition of adjoins} \\ &= \left\langle \sum_{i=1}^k \langle y, x_i \rangle e_i, e_j \right\rangle \\ &= \langle y, x_j \rangle\end{aligned}$$

This implies that $\Theta^* e_j = x_j$. Thus for each $y \in \mathcal{H}$, we have

$$\Theta^*\Theta y = \Theta^* \left(\sum_{i=1}^k \langle y, x_i \rangle e_i \right) = \sum_{i=1}^k \langle y, x_i \rangle \Theta^* e_i = \sum_{i=1}^k \langle y, x_i \rangle x_i$$

This motivates the definition of our last operator, the frame operator.

Definition The *frame operator* (S) is defined as $Sy = \Theta^*\Theta y = \sum_{i=1}^n \langle y, x_i \rangle x_i, \ \ y \in \mathcal{H}$

Note that we can find the frame operator for any collection of vectors, not just frames.

Using the reconstruction formula for tight frames, we find that for a tight frame, $S = AI$, where A is the frame bound.

1.2.2 Reconstructive Property of Parseval Frames

Frames are also useful because they have a reconstruction formula, similar to Proposition 1.1. For Parseval frames the reconstruction formula turns out to be the same as Proposition 1.1.

Proposition 1.3 ([2]). *$\{x_i\}_{i=1}^k$ is a Parseval frame for a Hilbert space $\mathcal{H}$ if and only if the following formula, called the reconstruction formula, holds for all $x \in \mathcal{H}$.*

$$x = \sum_{i=1}^k \langle x, x_i \rangle x_i$$

Proof. Assume that $\{x_i\}_{i=1}^k$ satisfies the reconstruction formula. We wish to show $\|x\|^2 = \sum_{i=1}^k |\langle x, x_i \rangle|^2$ (i.e. $\{x_i\}_{i=1}^k$ is a Parseval frame).

$$
\begin{aligned}
\|x\|^2 = \langle x, x \rangle = \left\langle \sum_{i=1}^{k} \langle x, x_i \rangle x_i, x \right\rangle \\
&= \sum_{i=1}^{k} \langle \langle x, x_i \rangle x_i, x \rangle \\
&= \sum_{i=1}^{k} \langle x, x_i \rangle \langle x_i, x \rangle \\
&= \sum_{i=1}^{k} | \langle x, x_i \rangle |^2
\end{aligned}
$$

To prove the converse, assume that $\{x_i\}_{i=1}^k$ is a Parseval frame. Note that $\|\Theta x\|^2 = \sum_{i=1}^k | \langle x, x_i \rangle |^2 = \|x\|^2$ for all $x \in \mathcal{H}$. This shows that the analysis operator (Θ) is an *isometry*. Since Θ is an isometry, we find that $\|\Theta x\| = \|x\|$ and $\langle \Theta x, \Theta y \rangle = \langle x, y \rangle$. Let $\{e_j\}_{i=1}^k$ be an orthonormal basis for $\mathcal{H}$.

$$
\begin{aligned}
x &= \sum_{j=1}^{n} \langle x, e_j \rangle e_j \\
&= \sum_{j=1}^{n} \langle \Theta x, \Theta e_j \rangle e_j \\
&= \sum_{j=1}^{n} \sum_{i=1}^{k} \langle x, x_i \rangle \overline{\langle e_j, x_i \rangle} e_j \\
&= \sum_{i=1}^{k} \langle x, x_i \rangle \sum_{j=1}^{n} \langle x_i, e_j \rangle e_j \\
&= \sum_{i=1}^{k} \langle x, x_i \rangle x_i
\end{aligned}
$$

□

2 Rank-One Decompositions

This section develops some of the tools we used to find our results in the next section. While the proof of Proposition 2.1 is challenging, the reader is encouraged to attempt it. A demonstrative example of the proof is also provided. However, if the reader wishes to read only the statement of Proposition 2.1 (and skip the proof), the reader will still be able to under stand the material in the next section.

Recall that a rank-one operator is a map whose range has dimension one. We can write every rank-one operator as $x \otimes y$, for $x, y \in \mathcal{H}$, where $(x \otimes y)w = \langle w, y \rangle x$. If x, y are in $\mathbb{R}^n$

or $\mathbb{C}^n$, the matrix representation of this operator is xy^*:

$$x \otimes y = xy^* = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} \bar{y_1} & \cdots & \bar{y_n} \end{bmatrix} = \begin{bmatrix} x_1\bar{y_1} & \cdots & x_1\bar{y_n} \\ \vdots & \ddots & \vdots \\ x_n\bar{y_1} & \cdots & x_n\bar{y_n} \end{bmatrix}$$

We then find that the frame operator $S = \Theta^*\Theta$ can be expressed as the sum of rank-one operators:

$$Sx = \sum_{i=1}^{k} \langle x, x_i \rangle x = \sum_{i=1}^{k} (x_i \otimes x_i)x$$

Also note if x_i is a unit vector, $x_i \otimes x_i$ is a rank-one projection. Thus to show that S is the frame operator for a unit tight frame all we need to show is that it can be expressed as the sum of rank-one projections.

Proposition 2.1 ([3]). *Let $\mathcal{H}$ be a Hilbert space with dimension n. A positive operator T on $\mathcal{H}$ can be written as a sum of k rank-one projections if and only if $k \geq$ rank(T) and the trace of T is equal to k.*

Note: In [3] the proof assumes that T is an invertible operator. For our purposes, we can not assume this. We have modified the proposition and its proof to suit our needs.

Proof. First we will prove the forward direction. Since the trace of a projection is equal to the dimension of its range, $\mathrm{tr}(x \otimes x) = 1$. Also, the trace of operators is additive. That is, $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$. Asumming that T can be written as the sum of k rank-one projections, we find

$$\mathrm{tr}(T) = \mathrm{tr}(x_1 \otimes x_1 + x_2 \otimes x_2 + \cdots + x_k \otimes x_k) = \mathrm{tr}(x_1 \otimes x_1) + \mathrm{tr}(x_2 \otimes x_2) + \cdots + \mathrm{tr}(x_k \otimes x_k) = k$$

Thus $\mathrm{tr}(T) = k$. Also, since T is the sum of k rank-one operators, the largest rank T can have is k. Thus $k \geq \mathrm{rank}(T)$.

To prove the converse, we will use induction on k to find unit vectors $x_1, \cdots, x_k$ such that T is the sum of the operators $x_i \otimes x_i$. The base case is when $k = n = 1$, where T itself is a rank-one projection and has trace 1. Thus we can write $T = x_1 \otimes x_1$.

The general idea is that we take T, and find a rank-one operator such that $T - x_k \otimes x_k$ is still a positive operator. We will then take this operator and find another $x_k \otimes x_k$ to subtract off. We will repeate this until we get a single rank-one projection. This then gives us how to express T as the sum of k rank-one projections. The details are as follows:

Since T is a positive (and thus self-adjoint) operator, we can find an orthonormal basis for T such that T is a diagonal matrix with positive entries $a_1 \geq a_2 \geq \cdots \geq a_n > 0$. There are two possibilites: $k > n$ or $k = n$. Since, by assumption, $k \geq \mathrm{rank}(T)$, if $k < n$ we find $\mathrm{rank}(T) < n$. Thus the diagonal representation of T includes some zero entries along the diagonal. We are only interested in the submatrix with positive entries. Thus this submatrix has dimention $\mathrm{rank}(T) = n$, and since $k \geq \mathrm{rank}(T)$ we find $k \geq n$. Thus this is really one of the previous two cases.

Case 1: $k > n$. Since $\text{tr}(T) = a_1 + a_2 + \cdots + a_n = k > n$ and $a_1 \geq a_2 \geq \cdots \geq a_n > 0$, we find $a_1 > 1$. Therefore we can let $x_k = e_1$, where $\{e_i\}_{i=1}^n$ is the standard orthonormal basis for $\mathcal{H}$. The operator $T - (x_k \otimes x_k)$ will have diagonal entries $a_1 - 1, a_2, \cdots, a_n \geq 0$, and thus is still a positive operator with rank n and trace $k - 1$.

Case 2: $k = n$. Since $\text{tr}(T) = a_1 + \cdots + a_n = k = n$, we know $a_1 \geq 1$ and $a_n \leq 1$. Define the function μ_n to give the n^{th} largest eigenvalue (counting multiplicity) of any self-adjoint operator on $\mathcal{H}$. For example, $\mu_n(T) = a_n$. Note that $\mu_n(T - (e_1 \otimes e_1)) \geq 0$ and $\mu_n(T - (e_n \otimes e_n) \leq 0$. Since $\mu_n(T - (x \otimes x))$ is a continuous function with respect to the vector x, we see by the Intermediate Value Theorem that there exists a unit vector y such that $\mu_n(T - (y \otimes y)) = 0$. Let $y = x_k$. The operator $T - (x_k \otimes x_k)$ then has rank $n - 1$ since it has 0 as its smallest eigenvalue. It also has trace of $k - 1 = n - 1$. Now we continue to work with the matrix with dimension $n - 1$. We repeat Case 2 till we get $n = k = 1$, which is our base case.

□

To help the reader grasp how this proof works, we provide a demonstrative example.

Example Let

$$T = \begin{bmatrix} \frac{5}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

be a linear operator on $\mathbb{R}^3$. Since $\text{tr}(T) = 5$ and $\text{rank}(T) = 3$, T can be written as a sum of 5 rank-one projections. Since $\text{tr}(T) > \text{rank}(T)$, we are in Case I, so we must find some x_5 such that $\text{tr}(T - x_5) = 4$ and $\text{rank}(T - x_5) = 3$. Let

$$x_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Then,

$$T - x_5 \otimes x_5 = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This new operator has trace 4 and rank 3, so we are still in Case I. We repeat the process; let

$$x_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now we have

$$T - x_4 \otimes x_4 - x_5 \otimes x_5 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This operator has trace 3 and rank 3. Now we are in Case II (and will remain in Case II). Thus, we need some vector x_3 such that the third largest eigenvalue of $T - \sum_{i=3}^{5} x_i \otimes x_i$ is zero. Let

$$x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Now we have

$$T - \sum_{i=3}^{5} x_i \otimes x_i = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\mathrm{tr}(T - \sum_{i=3}^{5} x_i \otimes x_i) = \mathrm{rank}(T - \sum_{i=3}^{5} x_i \otimes x_i) = 2$, so we find x_2 such that $\mathrm{tr}(T - \sum_{i=2}^{5} x_i \otimes x_i) = \mathrm{rank}(T - \sum_{i=2}^{5} x_i \otimes x_i) = 1$. Let

$$x_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

We now have

$$T - \sum_{i=2}^{5} x_i \otimes x_i = \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & 0 \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Finally,

$$\begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & 0 \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} = x_1 \otimes x_1$$

where

$$x_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

This is our base case, so the recursion stops here. Since $\|x_i\| = 1$ for all $\{x_i\}_{i=1}^{5}$, each $x_i \otimes x_i$ is a rank-one projection. Thus, we can write $T = \sum_{i=1}^{5} x_i \otimes x_i$ as a sum of rank-one projections.

Another tool we use in the next section that helps us study the trace of the frame operator is:

Lemma 2.2. *A frame operator $S = \Theta^* \Theta$ consisting of k unit vectors has trace k.*

Proof. The frame operator S can be expressed as $\sum_{i=1}^{k} x_i \otimes x_i$. When $\|x_i\| = 1$, $x_i \otimes x_i$ is a projection. Since the trace of a projection is equal to the dimention of the range, the trace of $x_i \otimes x_i$ is 1. Thus, for unit frames, $\mathrm{tr}(S) = \mathrm{tr}\left(\sum_{i=1}^{k} x_i \otimes x_i\right) = \sum_{i=1}^{k} \mathrm{tr}\left(x_i \otimes x_i\right) = \sum_{i=1}^{k} 1 = k$. $\square$

3 Surgeries

Definition A (k, l)-surgery removes k vectors from a collection of vectors (possibly a frame) and replaces them with l vectors. The remaining collection of vectors *must* be a tight frame. As long as there exists *at least* one collection of k vectors that can be removed and *at least* one collection of l vectors that can be added the surgery is possible. If no such collection of vectors exists, then the surgery is impossible.

3.1 Unit Tight Frames

3.1.1 $(0, m)$-Surgeries

Theorem 3.1. *Given a collection of unit vectors $\{x_i\}_{i=1}^k \subset \mathcal{H}$, a $(0, m)$-surgery resulting in a unit tight frame is possible if*

- *All the vectors added to $\{x_i\}_{i=1}^k$ are unit vectors*
- *$m = An - k$ where A is the frame bound of the resulting frame*
- *$A \geq A_{min}$ where A_{min} is the largest eigenvalue of the frame operator of $\{x_i\}_{i=1}^k$.*

Proof. Consider the frame operator, $S = \Theta^*\Theta$ for the collection $\{x_i\}_{i=1}^k \in \mathcal{H}$. Since $\|x_i\| = 1$, we can write S as the sum of rank-one projections. We wish to add vectors to the collection such that

$$S' = S + \sum_{i=1}^{m} y_i \otimes y_i \tag{1}$$

Since we want S' to be the frame operator for a unit tight frame, $S' = AI$ where A is the frame bound and $\|y_i\| = 1$. We can find a collection of vectors $\{y_i\}_{i=1}^m$ that satisfy the above equation if $AI - S$ has trace m and is a positive operator (by Proposition 2.1). Let $T = AI - S$. T is a positive operator when its eigenvalues are non-negaitve. To find its eigenvalues, we must set $\det(M - \lambda_t I) = 0$.

$$\det(T - \lambda_t I) = \det(AI - S - \lambda_t I) = \det((A - \lambda_t)I - S) = \det(\lambda_s I - S) = 0$$

Note that the right hand side is the form of the eignevalues of S. If the eigenvalues of S are λ_s, then we find

$$\lambda_s = A - \lambda_t$$

Note that S is symmetric. Thus, by the Spectral Theorem ([1]), S has n real eigenvalues (counting multiplicites). Since $\lambda_t = A - \lambda_s$ and λ_s is always real, we can pick an A such that $\lambda_t \geq 0$ for all eigenvalues. Thus there exists some minimum value A_{min} such that for all $A \geq A_{min}$, T is a positive operator.

Now, consider trace. Since trace is additive, $\text{tr}(T) = \text{tr}(AI - S) = A\text{tr}(I) - \text{tr}(S) = An - \text{tr}(S) = An - k$ (by Lemma 2.2). We want $\text{tr}(T) = m$, so we find $An - k = m$. Thus $k = An - m$. There exists some value of A such taht $A \geq A_{min}$ (ensuring that T is positive) and $A = \dfrac{k+m}{n}$ (where $m, n \in \mathbb{N}$, ensuring that $\text{tr}(T) = m$).

Since $AI - S$ is positive and has trace m, we can express $AI - S$ as the sum of m rank-one projections by Proposition 2.1. Thus it is always possible to form a tight frame by adding unit vectors to a collection of unit vectors.

Moreover, we can see from the trace equation that the number of vectors, m, that we need to add is $An - k$. Also, as noted earlier, the frame bound A must satisfy $A \geq A_{min}$ and A_{min} is the largest eigenvalue of S.

Example Consider the Mercedes-Benz Frame in Figure 1. Lets consider possible $(0, m)$-surgeries we can perform. Note that in we can not use the Parseval version of the Mercedes-Benz; we must make it a Unit Tight Frame to apply Theorem 3.1. Let $\mathcal{F} = \{x_1, x_2, x_3\}$ where

$$\{x_1, x_2, x_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \sqrt{\frac{3}{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\sqrt{\frac{3}{2}} \end{bmatrix} \right\}$$

The largest eigenvalue of the frame operator for $\mathcal{F}$ is $\frac{3}{2}$. Thus the number of vectors we need to add is $m = 2A - 3$, where $A \geq \frac{3}{2}$. For our example, lets suppose we wish to add 2 vectors to the Unit Mercedez-Benz. Then we find $A = \frac{5}{2}$.

To find the two additional unit vectors, we need to calculate the rank-one decomposition of the operator $AI - S$, where S is the frame operator for $\mathcal{F}$. We find

$$AI - S = \frac{5}{2}I - \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y_1 \otimes y_1 + y_2 \otimes y_2$$

where $y_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ and $y_2 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$. Note that there are other possible values for y_1 and y_2; this is just one possibilty.

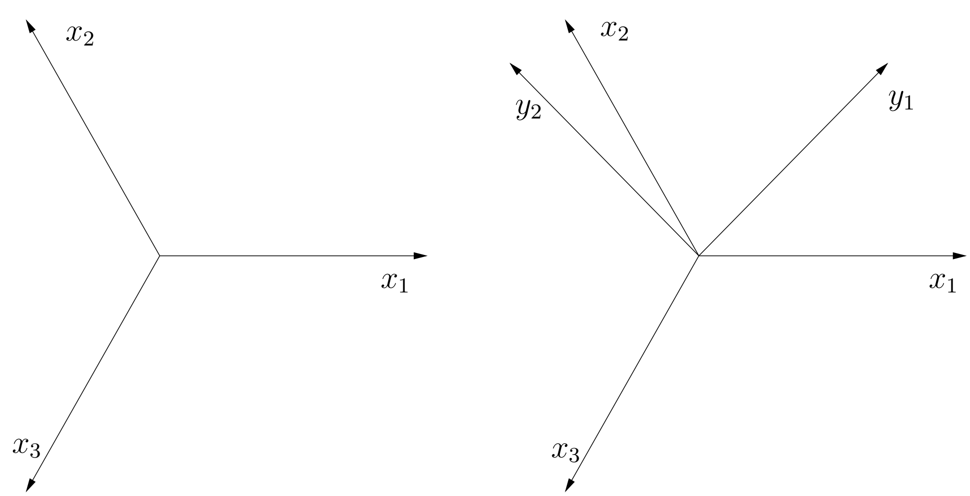

Figure 2: The Mercedes-Benz, before and after a (0,2) Surgery

$\square$

3.1.2 (l, m) Surgeries

Using Theorem 3.1, we can investigate questions about other surgeries on unit tight frames. First consider a $(1, k)$ surgery on a unit tight frame.

Proposition 3.2. *Given a unit tight frame, it is always possible to perform a $(1, t)$ surgery, where $t \geq 1$, and retain a unit tight frame.*

Proof. Consider the collection of vectors, $\{x_i\}_{i=1}^{k-1}$, formed by the unit tight frame with one vector removed. From Theorem 3.1, we know there exists some A_{min} such that when $A \geq A_{min}$, we can add $An - k$ vectors to form a tight frame. However, we know we can add one unit vector to the collection (and get the orriginal tight frame). Thus there exists an $A_1 \geq A_{min}$ such that $A_1 n - k = 1$. Thus, if we let $A_t = A_1 + (t-1)/n$, then $A_t n - k = (A_1 + (t-1)/n)n - k = A_1 n - k + (t-1) = 1 + (t-1) = t$. Since $A_t \geq A_{min}$ and $A_t n - k = t$, it is always possible to add $t \geq 1$ unit vectors to the collection of vectors $\{x_i\}_{i=1}^{k-1}$ and form a unit tight frame. In other words, it is always possible to do a $(1, t)$ surgery and retain a unit tight frame.

□

We now generalize this to (n, m) surgeries.

Theorem 3.3. *Given a unit tight frame, it is always possible to perform a (n, m) surgery and retain a unit tight frame if $m \geq n$.*

Proof. This proof is analogous to the proof of Proposition 3.2. □

3.2 Nonuniform Tight Frames

In the previous section we required that the surgeries created frame vectors with length one, giving us a unit tight frame. We did not put any restrictions of the frame bound of the resulting frame. In this section, we do not put any restrictions on the lengths of the vectors; we require that the frame bound of the original frame and frame resulting from the surgery be the same.

Theorem 3.4. *Let $\mathcal{F} = \{x_i\}_{i=1}^{k}$ be a tight frame for $\mathbb{C}^n$ with frame bound A.
Consider $\mathcal{G} = \{x_1, x_2, \ldots, x_{k-l}, y_1, y_2, \ldots, y_m\} \subset \mathbb{C}^n$. A (l, m)-surgery on $\mathcal{F}$, resulting in $\mathcal{G}$, is a tight frame for $\mathbb{C}^n$ with frame bound A only if*

$$\sum_{i=k-l+1}^{k} \|x_i\|^2 = \sum_{j=1}^{m} \|y_j\|^2$$

and

$$span\{x_i\}_{i=k-l+1}^{k} = span\{y_j\}_{j=1}^{m}$$

Proof. If $\mathcal{F}$ and $\mathcal{G}$ are both $\mathbb{C}^n$ tight frames with frame bound A, then both $\mathcal{F}$ and $\mathcal{G}$ have frame operator $S = AI$. So,

$$AI = \sum_{i=1}^{k} x_i \otimes x_i = \sum_{i=1}^{k-l} x_i \otimes x_i + \sum_{j=1}^{m} y_j \otimes y_j$$

$$\sum_{i=k-l+1}^{k} x_i \otimes x_i = \sum_{j=1}^{m} y_j \otimes y_j$$

$$\sum_{i=k-l+1}^{k} tr(x_i \otimes x_i) = \sum_{j=1}^{m} tr(y_j \otimes y_j)$$

$$\sum_{i=k-l+1}^{k} \|x_i\|^2 = \sum_{j=1}^{m} \|y_j\|^2$$

as $\text{tr}(x \otimes x) = \|x\|^2$ for all $x \in \mathbb{C}^n$.

To show that $\text{span}\{x_i\}_{i=k-l+1}^{k} = span\{y_j\}_{j=1}^{m}$, let

$$B = \sum_{i=k-l+1}^{k} x_i \otimes x_i = \sum_{j=1}^{m} y_j \otimes y_j$$

Let $\text{rank}(B) = d \leq n$. As B is positive, there exists a basis for $\mathbb{C}^n$ with respect to which B is an $n \times n$ diagonal matrix with entries $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > \lambda_{d+1} = \lambda_{d+2} = \cdots = \lambda_n = 0$ along its diagonal. The a^{th} entry along the diagonal of B is given by

$$\lambda_a = \sum_{i=k-l+1}^{k} x_{i(a)}^2 = \sum_{j=1}^{m} y_{j(a)}^2$$

where $x_{i(a)}$ and $y_{j(a)}$ are the a^{th} entries of x_i and y_j, respectively. As $\lambda_a = 0$ for $d < a \leq n$, $x_{i(a)} = y_{j(a)} = 0$ for $d < a \leq n$. Thus, dim $span\{x_i\}_{i=k-l+1}^{k} \leq d$ and dim $span\{y_j\}_{j=1}^{m} \leq d$. Since B is diagonal, there exist $\{w_h\}_{h=1}^{d} \subset \mathbb{C}^n$ such that $\{Bw_h\}_{h=1}^{d}$ is a linearly independent set. As $\{Bw_h\}_{h=1}^{d} \in span\{x_i\}_{i=k-l+1}^{k}$ and $\{Bw_h\}_{h=1}^{d} \in span\{y_j\}_{j=1}^{m}$,

$$span\{x_i\}_{i=k-l+1}^{k} = span\{Bw_h\}_{h=1}^{d} = span\{y_j\}_{j=1}^{m}$$

□

Now we consider a more specific case; a (1,2) surgery.

Lemma 3.5. *Let $\mathcal{F} = \{x_i\}_{i=1}^{k} \subset \mathbb{C}^n$ be a tight frame with frame bound A, and let $\mathcal{G} = \{x_1, x_2, \ldots, x_{k-1}, y_1, y_2\} \subset \mathbb{C}^n$ be the set resulting from a (1,2)-surgery on $\mathcal{F}$. $\mathcal{G}$ is a tight frame for $\mathbb{C}^n$ if and only if $y_1, y_2 \in span\{x_k\}$ and*

$$\|x_k\|^2 = \|y_1\|^2 + \|y_2\|^2$$

Proof. If $\|x_k\|^2 = \|y_1\|^2 + \|y_2\|^2$ and $y_1, y_2 \in span\{x_k\}$, then $span\{x_k\} = span\{y_1, y_2\}$. So, by the above theorem these are necessary conditions for $\mathcal{G}$ to be a tight frame for $\mathbb{C}^n$; thus, it remains to prove that these are sufficient conditions, i.e. that $x_k \otimes x_k = y_1 \otimes y_1 + y_2 \otimes y_2$. Assume that $\|x_k\|^2 = \|y_1\|^2 + \|y_2\|^2$ and $y_1, y_2 \in span\{x_k\}$. Then, there exist $r_1, r_2 \in \mathbb{R}$ such that, for $m = 1, 2, \ldots, n$, $y_{1(m)} = r_1 x_{k(m)}$ and $y_{2(m)} = r_2 x_{k(m)}$

$$\|x_k\|^2 = \|y_1\|^2 + \|y_2\|^2$$
$$\|x_k\|^2 = r_1^2\|x_k\|^2 + r_2^2\|x_k\|^2$$
$$r_1^2 + r_2^2 = 1$$

The (l, h) entry of the matrix form of $x \otimes x$ is $x_l x_h$. Thus, showing that $x_{k(l)}x_{k(h)} = y_{1(l)}y_{1(h)} + y_{2(l)}y_{2(h)}$ is equivalent to showing that $x_k \otimes x_k = y_1 \otimes y_1 + y_2 \otimes y_2$.

$$\begin{aligned} y_{1(l)}y_{1(h)} + y_{2(l)}y_{2(h)} &= r_1^2 x_{k(l)}x_{k(h)} + r_2^2 x_{k(l)}x_{k(h)} \\ &= (r_1^2 + r_2^2)x_{k(l)}x_{k(h)} \\ &= x_{k(l)}x_{k(h)} \end{aligned}$$

Thus $x_k \otimes x_k = y_1 \otimes y_1 + y_2 \otimes y_2$, and so $\mathcal{G}$ is a tight frame for $\mathbb{C}^n$ with frame bound A. ☐

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