CONSORTIUM

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From the Editor's Desk

JUST SAY NO (TO FILTERING)

Gary Froelich

A few weeks ago I read an Internet board posting by a woman whose accomplishments include the creation of an award-winning math Website for children.

She described herself as a very good math student who came to hate the subject while taking calculus in college. After finishing calculus, she decided she "would NEVER take another math course." She added that her son, an even better math student, developed a similar attitude while pursuing an engineering major.

About the legendary question, "Why do we have to learn this?" she wrote, "I think we need to take that question very, very seriously. The question is a signal that what is going on in our classrooms is perceived as irrelevant; and probably pretty dull stuff, too. If we can't justify it with any reasons better than, 'you'll need it for the next class' or '... for the test' or '... in case you become a grammarian/ mathematician/whatever,' then we need to seriously rethink what we are teaching and why."

I took her posting quite seriously; in part because I work for an organization whose goal it is to make experiences like those of the writer and her son uncommon. But the posting also struck home: like the writer, my daughter came to dislike math while taking calculus in college, and like the writer's son, my son began to dislike it while pursuing a technical major.

Many people seem unaware that mathematics is not taught solely because it is useful. If utility were the primary rationale, textbooks and teacher training programs would reflect the fact. There are other reasons. Years ago a mathematics professor told me, "Other departments use us as a cleaver." By that he meant that requiring a course that is abstract and irrelevant is a good way to thin the ranks. A student who did well in a relevant algebra course after failing one steeped in abstraction once observed, "Most people fail at something because they have no interest in it."

"A pump, not a filter" is a slogan heard throughout years of mathematics education reform. I believe that the filtering role's prominence can be measured by the extent to which instruction is bent on coverage and abstraction at the expense of relevance. In his book *The Arithmetic of Life and Death*, George Shaffner describes such instruction as having "too much abstraction, too much symbolism, too much complexity, too much rigor, and lessons that are too damned long."

If we are to reject the role of educational hit men (or women), then we must strive to engage our students, to make our courses relevant, and to help our students deepen their conceptual understanding of mathematics. This issue of *Consortium* carries writings of teachers and students from classrooms in which these goals are being realized. Moreover, at COMAP we are working hard on several new projects designed to help teachers achieve these goals. We invite our readers to watch *Consortium* and our Website for announcements and to get involved. ❏

Gary Froelich

CONSORTIUM

Consortium is a quarterly newsletter of the Consortium for Mathematics and Its Applications, Inc. (COMAP), but it is also much more. Each issue brings lessons and ideas that demonstrate what COMAP believes is an exciting way to teach and learn mathematics. The center Pull-Out Section of the issue is a classroom lesson ready to be photocopied and distributed to your students.

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WHY DOES A TRUCK SO OFTEN GET STUCK IN OUR UNDERPASS?

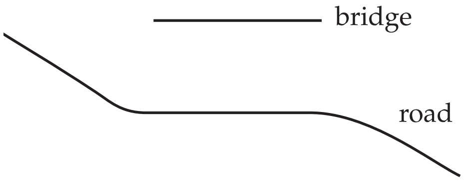
HENRY POLLAK

It just doesn't seem reasonable that trucks should have as much trouble as they do. After all, in every case I know, the road leading to the underpass has a sign indicating the clearance. Furthermore, it is natural to assume that the driver knows the height of the truck and is capable of comparing the two numbers and making an appropriate decision. And yet, the problem kept happening. I haven't kept any records, but I think they have decreased the clearance indicated on the sign at least twice in recent years. What's the matter, does the railroad bridge over the highway settle a little more every few years, or is the highway department just unable to hire anyone who can measure the distance between the pavement and the bottom of the bridge?

Surprisingly, I think it's neither of these: I think the trouble is that trucks have gotten longer!

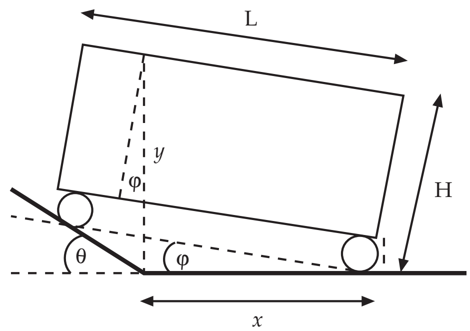
Henry Pollak spent 35 years at Bell Labs and Bellcore doing or managing mathematical research. He retired in 1986, and has been a visiting professor of mathematics education at Teachers College of Columbia University since 1987. This column will discuss mathematical questions and models which find their origin in either of his careers or in his everyday life. He can be reached at 40 Edgewood Road, Summit, NJ 07901-3988.

I live in Summit, New Jersey. The name of the community was not the fanciful invention of a developer's ad agency, but actually fits the geography: As a train on what used to be called the Delaware, Lackawanna, and Western RR makes its way west from the New York area, it has to go up several hundred feet to get here. In the old steam days, there was a watering station at which the engine could take a drink before continuing to Morristown and points northwest, and that may have contributed to the naming of the town. The road to which I have been referring goes directly down the side of the hill, while the track that crosses it changes elevation gradually along the edge of the same hill. If we draw a schematic of the height of the road, and of the bridge, it looks roughly as follows:



So as the truck goes down the hill, there comes a time when the cab and the front wheels are on the horizontal road surface under the bridge, while part of the van, and the rear wheels, are still on the hill. At this point, the middle of the van, being supported by the front wheels on the flat surface under the bridge and by the rear wheels still on the hillside, is further off the ground than the front of the van, and therefore closer to the bridge. It is indeed conceivable that the truck may get stuck under the bridge, even though its height is less than the height of the bridge. Nature has jacked up its rear end.

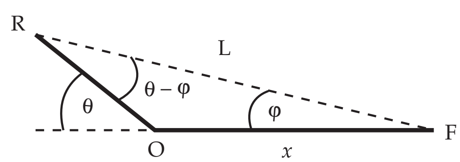
We need to draw a careful picture and introduce our notation. To simplify the model a little bit, without in any way endangering the reality of the problem, we assume that the junction between the flat road surface under the bridge and the sloping surface on the hillside is not curved and smooth as it is in the real world, but a corner. This changes the location of the truck for only a small portion of the roadway, and not where it makes any difference.



We see from the figure that y, the dangerous variable, is given by

$y = H \sec \varphi + x \tan \varphi.$

Because of the length of the truck, there is another relationship between x and φ. We redraw the bottom of the above figure, between the road surface and the line joining the bottom of the wheels:



By the Law of Sines in triangle ROF, we see that

$$\frac{\sin(\theta-\varphi)}{x} = \frac{\sin\theta}{L},$$

or

$x \sin \theta = L \sin (\theta - \varphi).$

This equation allows us to think of ϕ as our independent variable. When we substitute this for x in the equation for y, we obtain

$$\begin{aligned} y &= H \sec \varphi + L \frac{\sin(\theta-\varphi)}{\sin\theta} \tan \varphi \\ &= H \sec \varphi + L \frac{\sin\theta\cos\varphi - \cos\theta\sin\varphi}{\sin\theta} \tan \varphi \\ &= H \sec \varphi + L \sin \varphi - L \cot \theta \frac{\sin^2\varphi}{\cos\varphi}. \end{aligned}$$

Before proceeding further with the analysis, let us get an idea how big this effect might be.

Suppose that $H = 12$ feet, $L = 40$ feet, $\theta = 6°$, and $\varphi = 3°$. Then y, the clearance needed, would be 13.07 feet instead of 12 feet! That's a pretty big effect!

To find the worst possible value of φ, and hence x, we should take the above expression for y, differentiate it with respect to φ, and set the derivative equal to 0.

$$\frac{\partial y}{\partial \varphi} = H \sec \varphi \tan \varphi + L \cos \varphi - L \sin \varphi\,(2 + \tan^2 \varphi) \cot \theta.$$

When we set this equal to 0, we obtain a cubic equation for $\tan \phi$, namely

$\tan^3\varphi\,(L \cot \theta) + \tan \varphi\,(2L \cot \theta - H) = L.$

Since the angles are small, there is very little to be gained by attempting various forms of analytic trickery on this equation. If φ and θ are both small,

$\tan \varphi \approx \varphi, \cos \theta \approx 1, \sin \theta \approx \theta.$

We multiply the equation by $\sin \theta$, and the equation becomes

$\varphi^3 L + 2\varphi L - \varphi\theta H \approx L\theta.$

To first order, then, the φ that maximizes y will be given by $\varphi = \theta/2$. Then

$$y_{\max} \approx H + L\frac{\theta}{4}.$$

In our example, $\theta = 6° = \pi/30$ radians, and $L = 40$ feet. So the error of assuming the clearance is H is therefore $40\pi/120$ feet or about 1.05 feet. This matches the previous computation quite well.

In this problem, the computational process is perfectly stable. The source of the instability was the model itself: If you ignore the grade of the road, you may easily be off in the needed road clearance by over a foot! The effect just is not small, and if you ignore it, long trucks will indeed get stuck. ❏

THE UNIVERSALITY OF MATHEMATICS

(Part 2: The Distant Scene)

RICHARD FRANCIS

The focus of the story of mathematics in western culture begins essentially with the Dawn of History (the appearance of writing) and extends through the various Greek eras, the Dark Ages, the pre-Renaissance, and beyond. Such a partitioning of history raises questions about what was happening elsewhere in the world in key time intervals. What mathematical achievements are thus identifiable in non-western culture beginning with the Dawn of History and extending through the time period of the Renaissance?

Many questions arise as such a vast area of mathematics is explored. In what manner did the level of mathematical achievement in both western and eastern cultures correlate with plateaus of achievement in other areas of learning? Here the gamut of discussion ranges over technology, architecture, the sciences, and literature. Unfortunately, time and space do not permit such an ambitious undertaking. Only a scattered few can be considered as the story continues. The first concerns another look at the eastern setting, and then, beyond all of this, a brief glance at the mathematics of the New World.

JAPANESE MATHEMATICS

Some of the earliest traces of Japanese mathematics, corresponding to productive time periods elsewhere in the world, are lost to historians. There is nevertheless evidence of a strong influence by Chinese culture, especially in the sense of application, on the mathematics of Korea and Japan.

One of the more tangible evidences of this influence is the abacus, though significant variations on the remarkable calculating device are noticeable as one moves from one setting to another. Considerable interest was attached to the abacus, in Japan as elsewhere, because of its facility in dealing with money, weights, and measures. Thus its practical or applied features overrode the allied concerns of the development of mathematics as an academic discipline.

Japanese geometry, as is the universal custom, included a range of approximations to π. This was accompanied by basic formulas related to the circle and to certain polygonal types.

Little can be asserted with certainty concerning the state of Japanese mathematics in antiquity. Moving beyond the time period of the Dark Ages in western culture, a more intensive mathematical assessment becomes possible. Paralleling the Early Greek Era, names likewise emerged on the mathematical scene and thus allowed a more definitive look at achievement (as opposed to broad collective references).

Among the more notable figures of the early modern era was the mathematician Seki Kowa (1642–1708). Born in the same year as Isaac Newton, Kowa made his primary contributions in the discipline of algebra. Other areas of interest and research were those of the calendar, the solving of higher degree equations, and the solving of systems of equations.

It is in this latter area of achievement that Seki Kowa's name is best remembered. Working independently, his delvings into the solving of systems of equations led to the remarkable method of determinants. Names often associated with this development are those of Gabriel Cramer (1704–1752) and Gottfried Wilhelm Leibniz (1646–1716). Yet the discovery of determinants by Kowa likely occurred a full decade before that of the Leibniz discovery in 1693. The conventional restriction of determinants to solving systems of linear equations was expanded upon by Kowa so as to include extended systems of higher degree.

A Modern Notational Look at Kowa's Discovery.

If $\begin{matrix} ax + by = c \\ dx + ey = f \end{matrix}$, then

$$x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} = \frac{ce - bf}{ae - bd},$$

$$y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} = \frac{af - cd}{ae - bd}.$$

Though some notational and symbolic differences must be noted in the contrasting of western and eastern applications of determinant theory, counterparts are found in Kowa's work to the various row and column properties that are today very familiar to the mathematician. Property developments were motivated by the otherwise tedious techniques of determinant expansion in the absence of row and column relationships. Included were key properties such as

"The value of a determinant is unchanged if correspondingly numbered rows and columns are interchanged."

This insightful result led to the conclusion that all theorems relating to rows implied an equally valid theorem should the word "row" be replaced by the word "column." Moreover, the value of a determinant is zero provided any two of its rows (or of its columns) are identical. Or, the value of a determinant is unchanged if multiples of a given row (or column) are subtracted from the corresponding elements of another row (or column).

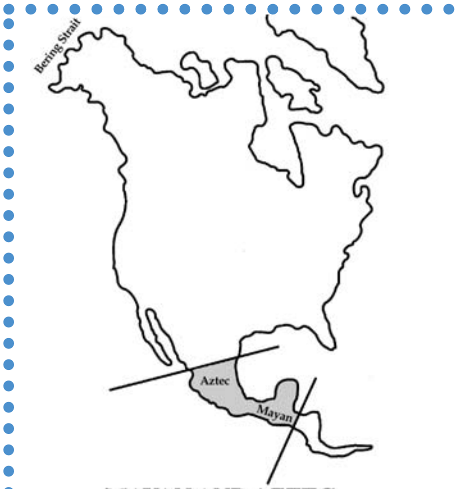
Seki Kowa, by profession a teacher of mathematics, did not venture much farther than the realm of algebra in his advancement of Japanese mathematics. However, those whom he taught and influenced were indeed to carry on with his example. As a consequence of the work of his students, including Takebe Kenko, these later steps led into a careful probing of the limit concept and the appearance in this Oriental setting of traces of the calculus. Such traces, though meager and not including the derivative, corresponded timewise rather closely with European enthusiasm over the newly formulated concepts of differentiation and integration in the Newton and Leibniz tradition.

EARLY MATHEMATICS OF THE NEW WORLD

Archaeological findings strongly imply the inhabiting of the American continents millennia before historic times. Such a pre-historic setting was, among varied possibilities, one of the migrating of Asian peoples across a then-existing Bering land bridge. Migration southward was likely very slow. Yet across the many centuries since this earliest crossing, the North American and South American continents were ultimately populated by widely scattered culture groups.

The long interval spanned times of wandering and then an eventual settling. Ways of living eventually changed and resulted in such practices as those of an agrarian and pastoral kind. Mathematical need was generally minimal in this long-ago primitive setting. However, civilizations did appear, especially those of Central America, for which records exist in some degree and thus provide a picture of early New World mathematical achievement. Archaeological findings suggest the appearance of migrants in this portion of Central America as early as 6000 B. C.

Among the most obvious of mathematical needs of the Mayans of the Yucatan were simple counting, the calendar, and possibly some determination of boundary lines (concerns quite reminiscent of the Egyptians in the Dawn of History setting). In such areas of basic necessity, the historian finds evidence of numerical and geometric understanding. This is further exemplified by an impressive architecture, much of which remains in the jungles of Central America to this day. Mayan civilization was at its pinnacle in the half millennium beginning with the fourth century (**Figure 1**).



MAYAN AND AZTEC GEOGRAPHICAL SETTING

FIGURE 1.

City-states were often formed based on language and cultural likenesses and thus gave distinctive markings to what is called Mayan mathematics. The more fundamental characteristics of Mayan mathematics are noted in their system of numeration, which was interestingly a vigesimal one and accordingly based on twenty. Such symbolism, composed largely of dots and bars, made provision for both place value and zero. In spite of the unusual number base, the system was conducive to the performing of the four fundamental operations and the extracting of certain roots. Evidence exists that suggests the actual performing of these more advanced operations by the Mayans.

The Mayan calendar, an impressive mathematical achievement, surpassed those of other New World civilizations of this early time period. In line with the vigesimal nature of their system of numeration, the year was partitioned into eighteen months of twenty days each and was accompanied by "leap year" modifications. Its features were so refined that the prediction of both solar and lunar eclipses became possible. Their architecture and engineering suggest traces of still more advanced mathematical capability. As the pre-Columbian era drew to a close, Mayan culture was in some state of decay and for many years following was given little historical acknowledgment concerning its technology, way of life, and mathematical advancements.

The Aztecs, also of the North American continent, had developed a form of writing as well as a system of numeration. Their system of numeration, based on twenty, likely stemmed from the number of one's fingers and toes and is thus an interesting variant on the digital concept. Such a system was of a relatively simple nature and was composed of varying pictures to denote key number values. Included was a scheme for the expression and use of fractional quantities. Application of these symbols was that of weights and measures, counting, astronomy, and a relatively accurate calendar. All of these areas of application gave evidence of their numeration system's vigesimal nature. An intermingling of astronomy and astrology implied a strong religious influence on Aztec mathematics. However, the numerological overtones in various calendar cycles do not preclude a degree of insight, though basic, into certain forms of geometry and trigonometry.

Unfortunately, records of Aztec civilization were destroyed in great measure. It was a loss which resulted in a disturbing gap in the historical record. Aztec culture flourished from the beginning of the thirteenth century to the times of Spanish conquest by Hernando Cortez in the early sixteenth century.

The Incan civilization of South America, without a written language, did produce a system of numeration (**Figure 2**). This system, in contrast to that of the Aztecs and the Mayans, was based on ten and allowed for simple counting and the most elementary forms of arithmetic. Though a reasonably accurate calendar was developed, it was oriented more toward agricultural concerns (the coming and going of the seasons) and did not reflect a detailed study of astronomy. Their engineering feats, especially of an architectural and bridge-building nature, suggest a more advanced knowledge of mathematics than that of numeration and simple counting. The most thriving time period of Incan civilization extended from the mid-fifteenth century to the mid-sixteenth century and encompassed a region of the South American continent ranging from Ecuador in the north to Chile in the south.

Early New World culture provides no remnants of what may be regarded as famous problems in mathematics. However, it does provide a piece in a worldwide puzzle that would otherwise be glaringly incomplete. The universal nature of mathematical interests and appropriate application are suggested, even here, in a geographical setting far removed from the mainstream of classical mathematical activity that has generally been the focus of the conventional historical look. Unfortunately, such a New World piece of the puzzle, because of the meagerness of records, still leaves unanswered a vast assortment of tantalizing questions.

FIGURE 2.

A CULTURAL BLENDING OF IDEAS

The intermingling of mathematical ideas dates from the earliest of time periods. Including commercial contacts involving Egypt, Crete, and Greece in the early Thalassic Age, or the spreading of Greek culture through the extensive military conquests of Alexander the Great, mathematical influences were overtly and subtly involved. The record gives abundant evidence of such periods of peaceful and violent interaction. Still at other times and places in history, a high degree of isolation dominated the scene.

Because of navigational contacts, Greek culture was significantly influenced by earlier civilizations of nearby lands which bordered on the Mediterranean as well as those somewhat to the east. Conjectures concerning discoveries in remotest antiquity of geometric relationships and their eventual assimilation into the mathematics of neighboring nations have long fascinated the curious as the historical record is examined. Not all discoveries have arisen independently, and thus raise the question of how other culture groups happened to become knowledgeable of such facts. The story is sometimes obscured by our inability to decipher long-lost languages. Many such encounters do not have the same positive outcome as that of the Rosetta Stone discovery.

As the varied cultures are studied, virtually without fail some form of numeration is noted. These numeration systems, vastly different in symbols and scheme, were often replaced by superior systems as awareness of the more sophisticated systems filtered through. The story of the transmission of Hindu-Arabic numerals into western culture is one of the most significant happenings of the pre-Renaissance.

Some forms of mathematical discovery do not have this feature of diverse finding or development. As noted before, early abstract mathematics makes virtually no appearance anywhere in the world except in the classical Greek environment. Hence, an appeal to other civilizations and culture groups (as in China or the Orient in general) sheds little light on the subject of how a form of demonstrative geometry is born.

Envision for a moment an alien visitor, looking for the first time at the state of earthly mathematical achievement. Think too in terms of the advanced mathematical awareness such an observer might possess. Would it not prove intriguing to witness the alien reaction to an assessment of the many centuries of worldwide attachment to Euclid's fifth postulate? This ancient, wordy pronouncement has a uniqueness that arouses curiosity. One would accordingly have to wonder how the notions of the point, the line, the plane, and of space itself, arose in the alien's far-off setting. Thus employing the Greek word for "world" instead of the inappropriate "geo" for "earth," the question becomes one of "kosomonometry" and its evolution on a distant but enlightened planet. Such reflections go beyond the bounds of the conventional question of the earthly origin and development of mathematical concepts. Still, the question proves insightful in the broader, more fanciful setting.

Famous problems of varying kinds, affording a meaningful perspective of history, often appear in the same unique manner of postulational geometry. These include, among other things, one of the earliest of such problem types, namely, the three famous problems of antiquity. Still others, as in the solving of Diophantine equations, suggest variants on the theorem of Fermat in scattered places. Even as traces of famous problems can often be found in diverse cultures, the matter of inductive speculation as opposed to rigorous proof must also be taken into account. How might, for example, an early disposition of the "Pythagorean Theorem" differ as the historical scene shifts from Egypt to China to Greece?

The picture, across the millennia, is paradoxically one of great contrast and similarity. However, the later years of the modern era are decidedly less fragmented in this respect. The mathematical community of the early twenty-first century is a worldwide community and does not present a picture of the isolation and accidental intermingling of ideas that so distinguished the earlier years. ❑

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Historical Notes concerns insights that focus on the history of mathematics. The chronological setting extends from that of a remote time period to the present day. Highlighted in such accounts are the varied perspectives relating to concepts, time periods, and mathematicians, be they well-known or otherwise. Historical material that lends itself to classroom use so as to instill mathematical appreciation within the student is especially appropriate.

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GETTING A BETTER ANGLE

Jonathan Choate

Since *Consortium* now contains lots of interesting material on modeling, it is time to close the Modeler's Corner and start a new column. As some of you may have noticed, many of my columns have had a distinctive geometric flavor so it should come as no surprise that the new column will be called Geometer's Corner. The goal of the column is to provide all of you who teach, geometry material that deals with topics that are not traditionally taught but that may have a place in the geometry classroom of the 21st century. I welcome any and all suggestions for future columns. I can be reached at jchoate@groton.org.

This month's column deals with one of my favorite problems. I originally found it in Heinrich Dorrie's *100 Great Problems in Elementary Mathematics* [1], a wonderful book now out in paperback that I highly recommend. The problem is often referred to as the Problem of Regiomontanus and is reputed to be the first max-min problem of the modern age.

> "At what point on the ground does a perpendicularly suspended rod appear largest [i.e., subtends the greatest visual angle]? It has been claimed that this was the first extreme problem in the history of mathematics since antiquity."[i]

A version of this problem is found in many calculus textbooks. The following version comes from Anton's *Calculus* [2].

> The lower edge of a painting, 10 ft. in height, is 2 feet above an observer's eye level. Assuming that the best view is obtained when the angle subtended at the observer's eye by the painting is maximum, how far from the wall should the observer stand?[ii]

I use a problem similar to this in my geometry class. I first introduce it when we begin to study right angle trig. All my students have TI-89 graphing calculators and they have learned how to use the inverse trig functions to find angles. To start we assume that the painting is 6 feet tall and is hanging in such a way that the bottom of the picture is 4 feet above an observer's eye level, which is 5 feet above the ground.

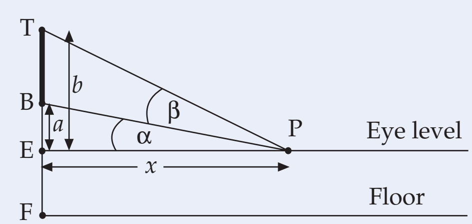


Figure 1. Regiomontanus's Viewing Angle Problem.

Figure 1 illustrates the problem. The hanging picture is represented by segment TB, and the viewing angle is represented by ∠BPT. PE represents the distance from the wall. At this point, you can have students calculate ∠BPT = ∠EPT – ∠EPB for different values of PE by using the formulas

$$\angle EPT = \tan^{-1}\left(\frac{10}{PE}\right) \text{ and } \angle EPB = \tan^{-1}\frac{4}{PE}.$$

If they put their results in table form they should get a table that looks like the one shown in **Table 1.**

PE	$\tan^{-1}\left(\frac{10}{PE}\right)$	$\tan^{-1}\frac{4}{PE}$	∠BPT
1	84.2895	75.9638	8.3257
2	78.6901	63.4350	15.2551
3	73.3008	53.1301	20.1707
4	68.1986	45.0000	23.1986
5	63.4350	38.6598	24.7752
6	59.0363	33.6901	25.3462
7	55.0080	29.7449	25.2631
8	51.3402	26.5651	24.7752
9	48.0128	23.9625	24.0503
10	41.9872	21.8014	20.1858

TABLE 1. VALUES FOR ∠BPT.

Once they have filled in the table have them plot the graph of ∠BPT in terms of PE. The graph gives some interesting information about the problem.

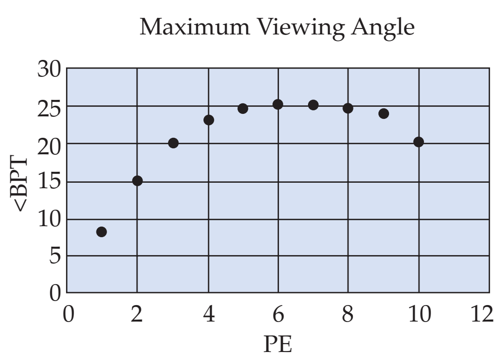


FIGURE 2. THE GRAPH OF ∠BPT AS A FUNCTION OF PE.

Hopefully they will see that the graph shows that there might be a maximum value for ∠BPT. Have them find this value to the nearest 1/100th using their calculators and a guess-and-check method of their choosing.

If you have access to a geometric construction program such as Cabri or Geometer's Sketchpad, you could use it to find another approximation by creating a sketch like the one in Figure 1 with P being a movable point. Once you have an approximation, ask students to convince themselves that for any value of the viewing angle less then the maximum there are two places where the viewer could stand and achieve that angle. They can argue this both by using the graph they created earlier or by using the sketch they created with the movable point P. For example, in Table 1 it shows that ∠BPT = 24.57772° for EP = 5 and EP = 8. At this point, they haven't found the exact solution but they do know that (a) one exists and (b) for any value less than the maximum there are two places the viewer can stand.

There are two non-calculus ways to find the exact solution, depending on what course you are teaching. In a geometry course I'd return to the problem when you begin a study of circles. Let's come back to the circles solution later. In a trig course you can continue with an algebraic solution such as the one given in Eli Maor's wonderful book *Trigonometric Delights* [3], which, thanks to the generosity of the good people at the Princeton Press, is available for free on the Internet. Here is Maor's solution[iii], which makes use of the fact that the arithmetic mean of two numbers is always greater than or equal to the geometric mean. This means that for all $u, v, >0$, $\frac{u+v}{2} \geq \sqrt{uv}$. Maor notes that to find the max value for the viewing angle ∠BPT one can look for the max value of tan(∠BPT) or the minimum value of cot(∠BPT). He wisely chooses for algebraic reasons to go with minimizing cot(∠BPT).

Here is his solution.

In what follows b = TE, a = BE, β = ∠EPT, α = ∠EPB, θ = ∠BPT = $\beta - \alpha$, and x = PE (see Figure 1).

Now, $\cot(\beta) = \frac{x}{b}$, $\cot(\alpha) = \frac{x}{a}$, and $\cot(\theta) = \cot(\beta - \alpha)$.

Using the formula for $\cot(\beta - \alpha)$ and making some substitutions, you get

$$\cot(\beta - \alpha) = \frac{\cot(\alpha)\cot(\beta)+1}{\cot(\alpha)-\cot(\beta)}$$

$$= \frac{\left(\frac{x}{a}\right)\left(\frac{x}{b}\right)+1}{\frac{x}{a}-\frac{x}{b}}$$

$$= \frac{x}{b-a} + \frac{ab}{(b-a)x}$$

Here is where the relationship between the arithmetic and geometric mean of two numbers comes in.

Let $u = \frac{x}{b-a}$ and $v = \frac{ab}{(b-a)x}$. Since $\frac{u+v}{2} \geq \sqrt{uv}$, we have

$$\frac{x}{b-a} + \frac{ab}{(b-a)x} \geq 2\sqrt{\left(\frac{x}{b-a}\right)\left(\frac{ab}{(b-a)x}\right)} = 2\frac{\sqrt{ab}}{b-a}.$$

There is equality when $u = v$ or when $\frac{x}{b-a} = \frac{ab}{(b-a)x}$. This implies that $x^2 = ab$ and finally that $x = \sqrt{ab}$.

So the exact solution to our original problem is that the person would have to stand $\sqrt{40}$ feet to get the maximum viewing angle. Neat—no calculus—just some trig and some clever algebra.

If you are introducing the problem in a geometry class, here is a geometric solution that requires some knowledge about inscribed angles in a circle and intersecting secant lines. Earlier, we saw that given an angle less than the maximum angle one could always find two points, call them P_1 and P_2 such that $\angle BP_1T = \angle BP_2T$. This is illustrated in **Figure 3.**

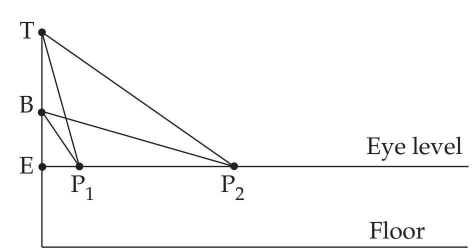


FIGURE 3. $\angle BP_1T = \angle BP_2T$

There is something special about points B, T, P_1 and P_2 that will jump out at you if you think of the segment BT as being a chord of a circle and $\angle BP_1T$ and $\angle BP_2T$ as being inscribed angles in a circle. All four points lie on a circle! If P is the point where the maximum occurs, then the circle through B and T will intercept the eye level line in only one place and hence it must be tangent at that point. Now the picture looks like **Figure 4.**

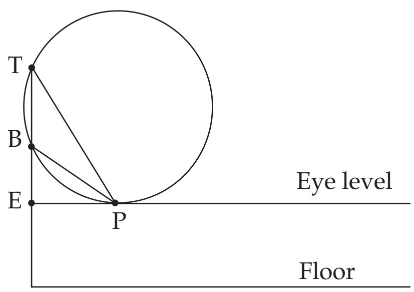


FIGURE 4. THE EXACT SOLUTION TO THE VIEWING ANGLE PROBLEM.

There is a theorem that relates secant lines to tangent lines, which says in this case that $TE \cdot BE = EP^2$. In our example, TE = 10, BE = 4 so EP = $\sqrt{40}$, the same answer we got analytically.

Let's go a bit farther with the problem and come up with a way to construct the circle tangent to the eye level line using only Euclidian tools. Here is one way of doing it.

Locate a point Q on line BE such that QE = BE and Q-E-B.

Construct the midpoint M of segment TQ.

Construct the circle with center M that passes through point T.

Construct a perpendicular to TQ through E and label its intersections with circle M as S_1 and S_2.

S_1 is the desired point of tangency.

Construct the circle C_2 through T, B and S_1.

Note that ΔQS_1T is a right triangle and S_1E is the altitude to its hypotenuse. Since the altitude to the hypotenuse is the geometric mean of the two segments into which it divides the hypotenuse you can show that $S_1E^2 = QE \cdot TE$. S_1 is the desired point of tangency.

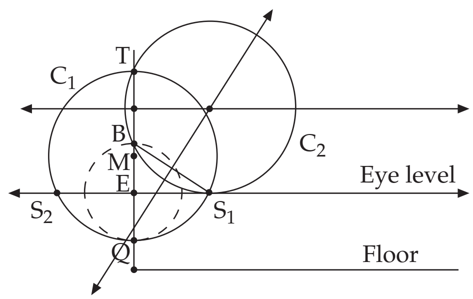


FIGURE 5. GEOMETRIC CONSTRUCTION OF MAXIMUM VIEWING ANGLE.

An interesting variation to the Art Gallery Problem is finding the best place to sit in a movie theatre with raked seating. I found the following version on the Grand Valley State University Mathematics Department Website located at www.gvsu.edu/math/calculus/M201/pdf/movie.pdf.

A movie theatre has a screen that is positioned at 10 feet off the floor and is 25 feet high. The first row of seats is placed 9 feet from the screen and the rows are 3 feet apart. The floor of the seating area is inclined at an angle of 20 degrees above the horizontal. Suppose your eyes are 4 feet above the floor and you want to locate a seat that gives you the maximum viewing angle. How far up the inclined floor should you locate the seat?[iv]

In the **Figure 6,** TB = 25, BG = 10, GA = 9, $\angle DAE = 20°$ and CA = 4 and line PH is parallel to line AD. This problem has a similar constructive solution to that of the art gallery. You need to find a point P such that the circle through T, B, and P is tangent to line PH. In order to do this, you need to construct the point H, which is where the line that determines your eye level intersects the wall. Once you have that point located you need to find the length of HP, the geometric mean of TH and BH. The length of segment CP is how far you should place your seat up the raked floor.

Using the law of cosines, you can find an expression for $\angle BPT$. Start with a coordinate system with origin at G. In this system T = (0, 35), B = (0, 10), and A = (9, 0). Since line AD has slope tan(20°) and goes through the point (9, 0), its equation in point-slope form is $y = \tan(20°)(x - 9)$. Line PH is parallel to line AD and is 4 units above it so its equation is $y - 4 = \tan(20°)(x - 9)$. Therefore, any point on line PH has coordinates $(x, \tan(20°)x - 9\tan(20°) + 4)$.

Expressing BP and TP in terms of x gives you

$$BP = \sqrt{x^2 + \left(10 - \left(\tan\left(20^\circ\right)x - 9\tan\left(20^\circ\right) + 4\right)\right)^2}$$

$$TP = \sqrt{x^2 + \left(35 - \left(\tan\left(20^\circ\right)x - 9\tan\left(20^\circ\right) + 4\right)\right)^2}$$

$\angle BPT$ can now be found using the Law of Cosines.

$$\angle BPT = \cos^{-1}\left(\frac{BT^2 + TP^2 - 625}{2BT \cdot PT}\right)$$

A maximum value for this function can now be found using a graphing calculator with a maximum function or by calculus. Once you have the

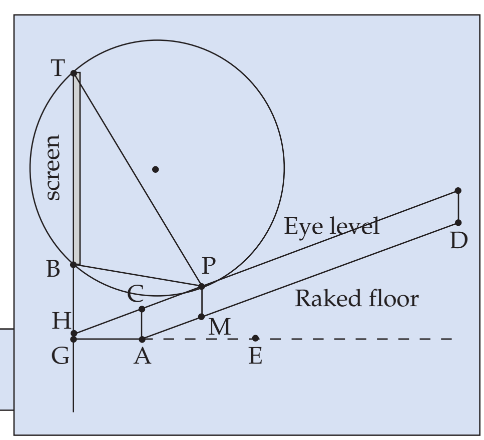


FIGURE 6. MAXIMUM VIEWING ANGLE IN A MOVIE THEATRE WITH RAKED SEATING.

coordinates of P, the length AM can be found, and to the nearest hundredth it is 8.25. Another solution using calculus can be found at www.gvsu.edu/math/calculus/M201/pdf/movie.pdf. This solution is derived from UMAP Module 729, "Calculus in a Movie Theatre."

It is interesting that both variations of Regiomontanus's problem can be solved exactly by construction. Both require you to do the following:

Given two points A and B and a line that does not contain A or B and that does not intersect segment AB, construct the circle that passes through A and B which is tangent to the given line.

I leave it to the interested reader to develop a Cabri or Geometer's Sketchpad solution to the movie theatre problem using a construction similar to the one used to find the solution to the Art Gallery problem.

Regiomontanus's original problem about the suspended rod can also be solved using the following construction:

Given two points A and B and a circle with center C such that A and B are external to C; A, B and C are collinear and A-B-C, construct the circle through A and B that is tangent to circle C.

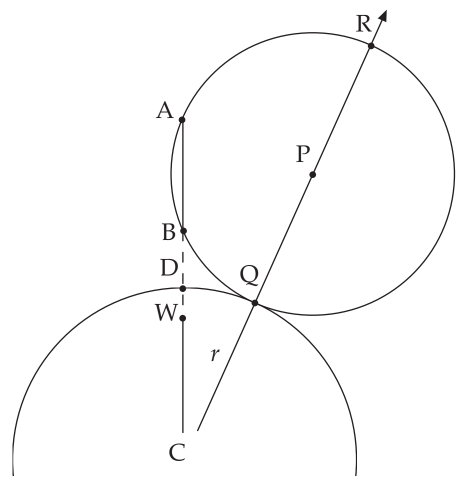
Here is a construction using Geometer's SketchPad. In **Figure 7,** circle P is tangent to circle C at Q. Q, C, and P are collinear, and CP extended meets circle P at R. Since secants AB and QR intersect at C, AC · BC = RC · QC. AC, BC, and CQ are known. If we let CQ = r and QP = s, then RQ = 2s, and RC = 2s + r. Therefore,

$$AC \cdot BC = (2s + r)r$$

$$= 2sr + r^2$$

and

$$s = \frac{AC \cdot BC - r^2}{2r}$$



AB = 12.2771 cm
BC = 8.3759 cm
r = 6.2475 cm

$$\frac{AB \cdot BC - r^2}{2r} = 5.1061 \text{ cm}$$

$$\frac{\frac{AB \cdot BC - r^2}{2r}}{r} = 0.82$$

FIGURE 7. SOLUTION OF REGIOMOTANUS'S ORIGINAL PROBLEM.

Here is how to do the construction shown in **Figure 7.**

Step 1. Measure AC, BC, and CD = r, the radius of the given circle.

Step 2. Use the SketchPad's calculator to calculate $s = \frac{AC \cdot BC - r^2}{2r}$

Step 3. With C as center dilate point D by a scale factor of s/r, creating a point W. Construct segment CW. Note that CW has length s.

Step 4. With A as center and CW as radius construct circle A. With B as center and CW as radius construct circle B. Label one of the points of intersection of the two circles P.

Step 5. Construct a circle with P as center that contains point A. This is the circle tangent to circle C.

Step 6. Label the intersection of circle P with circle C, Q.

∠AZB is the maximum angle!

If you would like a challenge, try the following.

Given a circle C and a line l that does not intersect C, let A and B be any two points on l. Find the point P on circle C such that ∠APB is a maximum.

If you come up with a solution please send it to me and I'll publish it in the next Geometer's Corner. ❑

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[i] Dorrie, page 369

[ii] Anton, page 499

[iii] Maor, page 46

[iv] GVSU web site, page 1

Send solutions to old problems and any new ideas to the Geometer's Corner editor: Jonathan Choate, Groton School, Box 991, Groton, MA 01450.

ONE EGG OR TWO?

STATISTICS HELPS SHED LIGHT ON PALEONTOLOGY MYSTERY

PAUL KEHLE & TED HODGSON

One of the goals of paleontology is to connect what we learn through fossil records of extinct animals' biology to the biology of contemporary animals. A product of such work is the construction of phylogenetic trees that trace animal morphology and behavior across many generations and species of animals. The detail in which the most recent branches in such trees can be articulated often overwhelms the detail possible for the oldest branches. In contrast to biologists who enjoy easy access to large numbers of living members of the species they study, paleontologists are at the mercy of what is originally recorded in the fossil record and what remains intact long enough to surface for examination. Therefore, the excitement generated by the recent finding of just one very well preserved nest of a small dinosaur in Montana, *Troodon formosus*, is understandable. Equally understandable is the paleontologists' desire to extract as much information from the nest as possible. As we will see, despite their rather paltry n value of 1, their quest led mathematicians to one of the limits of contemporary mathematics.

Troodon, Reptile, and Avian Biology

Although modern day birds and reptiles are very different species, it is quite likely that they share common ancestors. Anyone who has seen *Jurassic Park* probably remembers the closing scene of a pelican flying across the ocean's surface, suggesting that modern descendants of extinct dinosaurs are alive and doing quite well. The tracing of contemporary animals to their ancient ancestors makes use of every bit of evidence available in living and fossilized samples.

Modern day birds form and lay their eggs one at a time, with some time passing (perhaps a day or so) between the laying of successive eggs belonging to the same clutch. Reptiles lay their eggs all at once in a large depositing of a complete clutch. Avian ancestors also laid their eggs sequentially, and there is some reason to suspect that some of these ancestors had two ovaries functioning in tandem rather than the single ovary found in modern birds. Animals with two ovaries functioning in tandem will lay one pair of eggs at a time followed by a short interval before laying another pair. In trying to trace development of modern species, traits such as nesting behavior, egg formation, method of egg deposition, and number of ovaries play crucial roles.

As you might have guessed given the find of an intact *Troodon* nest—with eggs in place—the egg laying dynamics of *Troodon formosus* gained prominence with paleontologists trying to situate *Troodons* among the ancestry of modern birds and reptiles. **Figure 1** shows a photo of the nest and its eggs as fossilized in Montana about 75 million years ago. It contains 22 eggs.

FIGURE 1. PHOTOGRAPH OF A FOSSILIZED *TROODON* NEST FOUND IN MONTANA. THE DARKER OBLONG SHAPES ARE EGGS. CAN YOU FIND 22 OF THEM? PHOTO COUTESY OF MUSEUM OF THE ROCKIES.

The Paleontological Question

Without going into more of the biology of avian and reptile reproduction, we can focus on the question that led the paleontologists to confer with statisticians. Consider for a moment the four computer generated *Troodon* nests in **Figure 2.** Which egg arrangements do you think were produced by an animal with two ovaries operating in tandem, and which egg arrangements do you think were laid by an animal laying an entire clutch all at once?

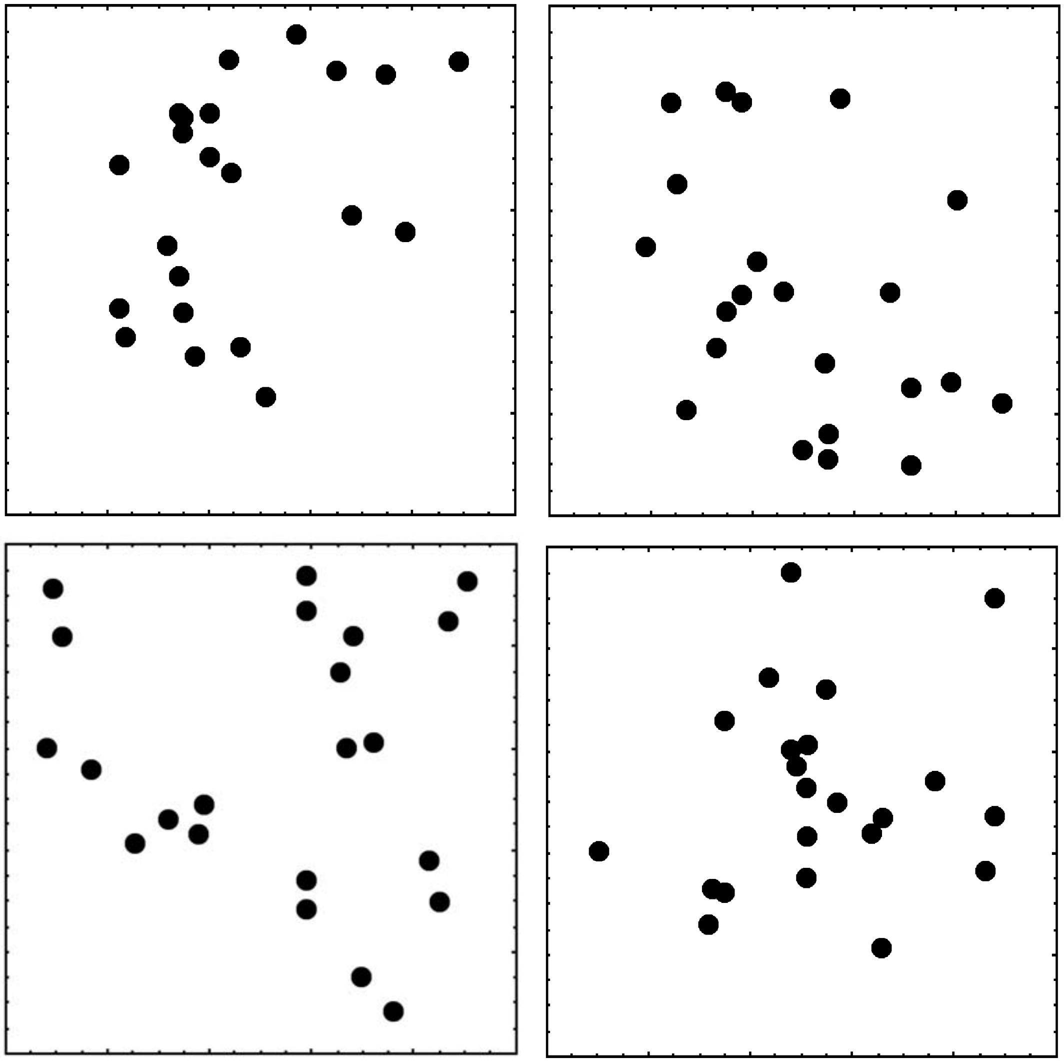

FIGURE 2. Four computer-generated nests and egg arrangements. Which were generated by a paired egg mechanism, and which were generated by a completely random mechanism?

Unfortunately the fossilized *Toodon* nest found in Montana was not as clear an example of an egg arrangement produced by a paired mechanism as is the simulated nest in the lower left corner of Figure 2. When confronted with an egg arrangement whose origin is uncertain, it is natural to turn to statistics to assign a degree of confidence to the question of whether the nest was produced via a paired egg laying mechanism or by a more random approach.

It is common for mathematics teachers and students, when posed with the problem of discriminating between paired and non-paired laying mechanisms, to quickly focus on a comparison of intra-pair distances with inter-pair separations. The intuition is that if the average of the intra-pair distances is small compared to the average of the inter-pair distances, then a paired mechanism was probably at work. Similarly, if the average of the intra-pair distances is comparable to the average of the inter-pair distances, then a paired mechanism was likely not present.

But this plan is well suited only when the likely pairs are easily identified. In some of the simulated nests in Figure 2, we would be hard pressed to pair the eggs up with any confidence at all. This inability, however, does not mean that the eggs were not laid in pairs. There is a region of overlap in the possible distributions of egg arrangements where nests produced by a paired mechanism and nests produced by a more random process look very similar.

In the in-between scenarios, where it is difficult to decide if a paired or random approach was used, we seem in need of a foothold or something to use as a basis of comparison. The statisticians provided just such a basis.

The Statistical Answer

We should first state some of the assumptions we are making before delving into the mathematical model used to answer the paleontologists' question. We are assuming that the eggs were not significantly displaced from their initial resting spots after being laid, we are assuming that we can treat the eggs as points in a plane rather than paying attention to the full three-dimensional aspect of the problem (in fact the *Troodon* nest was very flat and no eggs were piled on top of one another), and when we speak of a paired versus random mechanism we are not saying that there wasn't a random aspect present in the paired mechanism. In regard to this last assumption, the key idea is that despite the random locations of the pairs themselves, the paired nature (the intra-pair distances) was constrained by the animal not moving much between depositing the two eggs that belong to a pair.

So, where do we begin to assign a level of confidence to the hypothesis that the *Toodon* eggs were laid via a paired mechanism? Although paleontologists might be constrained by a very limited number of fossilized nests, mathematicians face no such constraints. In the abstract world of mathematics it is often easy to generate a hundred, a hundred thousand, or a hundred million simulated nests. With large numbers like these, we gain statistical power and confidence not possible when studying only one physical nest.

The next key simplifying assumption made by the statisticians was to basically ignore the problem of measuring the inter-pair distances. They sought only to compare the observed intra-pair distances in the fossil nest with an appropriate statistical distribution of intra-pair distances for simulated nests. But this left them with the problem of figuring out how to pair up the eggs in distributions where the pairs are not obvious. How could this be done?

The statisticians defined a new measurement associated with any distribution of points in the plane called the minimum paired distance, or MPD. Consider **Figure 3** which shows all possible pairings of a simulated nest of 6 eggs. The MPD for this distribution of eggs is the minimum sum (or one of the minimal sums in the event of two or more equivalent minima) of the intra-pair distances found across all 15 possible pairings.

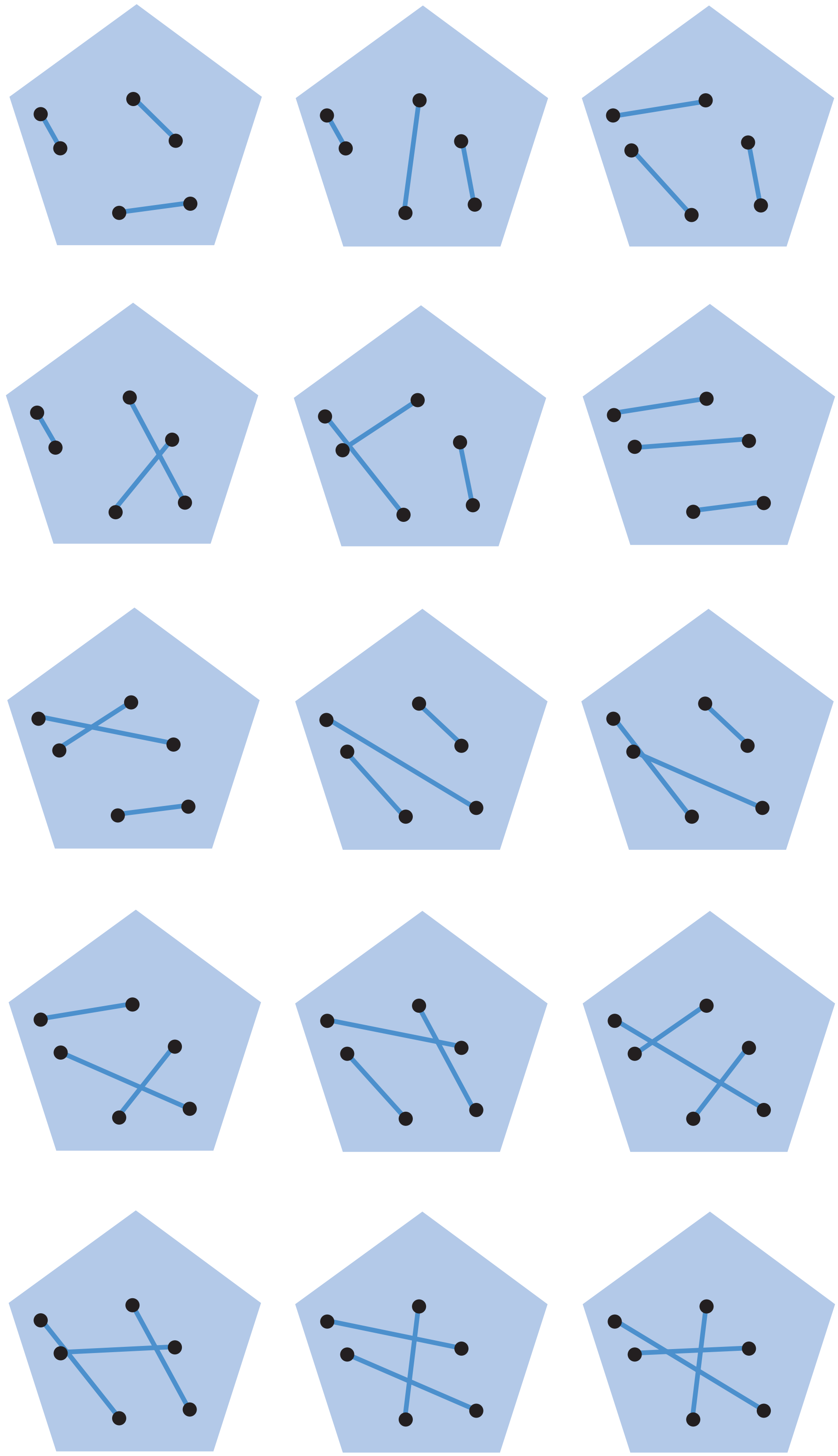

FIGURE 3. ALL 15 POSSIBLE WAYS OF PAIRING A SIMULATED NEST OF 6 EGGS. THE PAIRING THAT RESULTS IN THE SMALLEST SUM OF INTRA-PAIR DISTANCES IS THE MINIMUM PAIRING AND ITS ASSOCIATED SUM IS THE MINIMUM PAIRED DISTANCE, OR MPD.

Given the definition of MPD, we are now ready to calculate the distribution of MPDs for many simulated nests of 22 eggs. In generating these simulated nests, an important issue of scale must be addressed. The rough size and shape of *Troodon* nests, and the degree to which the eggs are distributed throughout the interior of the nests, are parameters that have to be set based on the observed nest. Once these are determined, the next step is to generate many simulated nests. The vital factor in generating these nests is that the generation be done randomly—with no pairing mechanism at work. This way we can compare the MPD for the fossil nest with the distribution of MPDs for the simulated nests and determine how significantly the fossil nest's MPD differs from the mean of the simulated nests' mean MPD.

Pause. Remember that when we go to write a program to generate the simulated nests and calculate their MPDs, we will have to first *find* the pairing of the randomly located 22 eggs that yields the MPD, and we will have to repeat this task for *each* nest. For 6 eggs there are 15 different pairings to consider. How many possible pairings are there for 22 eggs?

Combinatorial Detour

The question of how many pairings are possible given an even number of objects, n, is a nice challenge for students studying permutations and combinations. Both recursive and closed form expressions are possible and they seem to be generated equally depending only on a student's preferred way of thinking. The problem is challenging because after just the first couple of cases, the number of pairings becomes large quickly. For two objects, there is only one possible pairing, and for four objects there are three distinct pairings. As Figure 3 shows, there are 15 distinct pairings of six objects, and with the addition of just two more objects the number of pairings of eight objects jumps to 105. Caution: In what follows it will be important to keep straight the difference between a pair and a pairing: Each of the three possible *pairings* of four objects has two *pairs*; similarly, each of the 105 *pairings* of eight objects consists of four *pairs*.

Figure 4 shows how to build the group of pairings of six objects recursively out of the group of pairings of four objects. Remember that order within a pair or within a pairing doesn't matter.

If we let P_n represent the number of pairings of n (where n is any positive

Step 0. Given the three pairings of four objects:

AB CD

AC BD

AD BC

Step 1. Append to each pairing, the new pair EF.

AB CD EF

AC BD EF

AD BC EF

Step 2. Working in turn with each of these three new pairings, sequentially exchange one member of each pair with each of the members of the new pair to generate new pairings.

AB CD EF	→	AE CD BF	exchanging B and E
	→	AF CD EB	exchanging B and F
	→	AB CE DF	exchanging D and E
	→	AB CF ED	exchanging D and F
AC BD EF	→	AE BD CF	exchanging C and E
	→	AF BD EC	exchanging C and F
	→	AC BE DF	exchanging D and E
	→	AC BF ED	exchanging D and F
AD BC EF	→	AE BC DF	exchanging D and E
	→	AF BC ED	exchanging D and F
	→	AD BE CF	exchanging C and E
	→	AD BF EC	exchanging C and F

Figure 4. Generating the 15 pairings of 6 objects out of the 3 pairings of 4 objects.

even integer) objects we can develop a recursive formula for P_n by counting the numbers of exchanges made for each of the pairings in P_{n-2}. Consider any n, then there are $\frac{(n-2)}{2}$ pairs in each pairing of the $n - 2$ objects comprising the P_{n-2} pairings.

For example, if $n = 6$, then there are $\frac{(6-2)}{2} = 2$ pairs in each of the previous pairings of $6 - 2 = 4$ objects. For each of these pairs in just one pairing selected from the P_{n-2} pairings, we need to make *two* exchanges, once with each member of the new pair involved in moving from P_{n-2} pairings to P_n pairings. So, in step 2 in Figure 4, when we were working with the AD BC EF pairing, we had to make two exchanges in each AD and BC involving E and then F. This means we made $2 \cdot \frac{(n-2)}{2}$ new pairs or simply $n - 2$ new pairs. But we also made the very first new pair based on the previous AD BC pairing by appending EF to it. So in all we made $2 \cdot \frac{(n-2)}{2} + 1$ or $(n - 1)$ new pairs out of just one of the previous ($n = 4$) pairings. We repeat this process with each of the previous pairings for a total of P_{n-2} times. Hence, $P_n = (n - 1) \cdot P_{n-2}$. Knowing that $P_4 = 3$, for $n = 6$, we get $P_6 = 5 \cdot P_4$ or $P_6 = 15$. **Table 1** shows the first several values of P_n.

n	P_n
2	1
4	3
6	15
8	105
10	945
12	10395
14	135135
16	2027025
18	334459425
20	654729075
22	13749310575

TABLE 1. THE NUMBER OF DISTINCT PAIRINGS OF n OBJECTS.

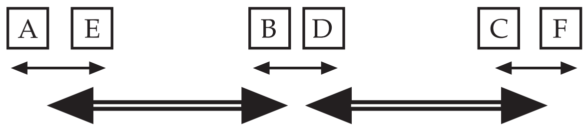

FIGURE 5. DERIVING P_n FROM THE $n!$ PERMUTATIONS OF n OBJECTS WHERE $n = 6$ BY TAKING INTO CONSIDERATION DIFFERENCES IN INTRA-PAIR AND INTER-PAIR ORDERINGS THAT DO NOT CHANGE THE OVERALL NATURE OF ANY ONE PAIRING.

It is also possible to derive a closed form expression for P_n by considering first all permutations of n objects and then removing the duplicate pairings due to intra-pair ordering and to inter-pair orderings.

Consider $n = 6$ objects that can be arranged in $n!$ ways in n boxes; and then consider any one of these pairings (e.g., AE BD CF). See **Figure 5.**

Each of the $\frac{n}{2}$ pairs within any given pairing could be in any of two possible orders (e.g., AE or EA), hence we must reduce the $n!$ permutations by a factor of $2^{n/2}$. Additionally, the $\frac{n}{2}$ pairs themselves could be in any of $\left(\frac{n}{2}\right)!$ orders. Applying both of these factors to $n!$ we get:

$$P_n = \frac{n!}{2^{\frac{n}{2}} \cdot \left(\frac{n}{2}\right)!}$$

May I Have the Envelope Please?

Returning to the question of whether *Troodon* had one or two ovaries, we were getting ready to run a computer program to generate lots of simulated nests and calculate the MPD for each nest. We now know that for each simulated nest, we will have to calculate the sum of the intra-pair distances for each of the 13,749,310,575 possible pairings of 22 eggs! If we want to generate a distribution of MPDs for 22 eggs that will allow for a high level of confidence we will want to look at thousands of simulated nests. The statisticians in Montana decided to use 1000 nests in part because of the time required to compute the needed 1.37×10^{13} intra-pair distance sums. In fact, they were able to avoid examining some of the pairings least likely to yield the MPD, but the problem remained computationally very intensive.

The graphs in **Figure 6** show the development of the increasingly normal distribution of MPDs for simulated nests with 6 eggs.

After the statisticians generated their 1000 simulated nests of 22 randomly distributed eggs and then compared the MPD of the fossilized nest (requiring another perusal of 13,749,310,575 possible pairings) to this distribution, they found it was located far to the left of the distribution. In fact, the MPD of the fossilized *Troodon* nest was smaller than 99% of the simulated MPDs. On this basis, the paleontologists concluded with 99% certainty that the *Troodon* eggs were laid in pairs lending support to the conjecture that they had two ovaries which links them with the ancestors of modern day birds.

Remaining Mathematical Questions and Challenges

In past *Math Today* columns we've looked at problems that become analytically and or computationally too challenging to solve. In this case, we saw how statisticians were able to answer a significant question that was at the border of what is computationally feasible. They were fortunate that the fossilized nest contained only 22 eggs. They might still be waiting for their computer programs to finish running if the paleontologists had brought them a nest of 30 eggs. To find the MPD for just one nest of 30 eggs would require looking at 6,190,283,353,629,375 distinct pairings, or 450,225 times as many pairings as are possible among 22 eggs.

Is there a better way? In particular is there a way to reduce the number of pairings we must search through to find the pairing that yields the MPD? Turn your students loose and see what they discover. ❑

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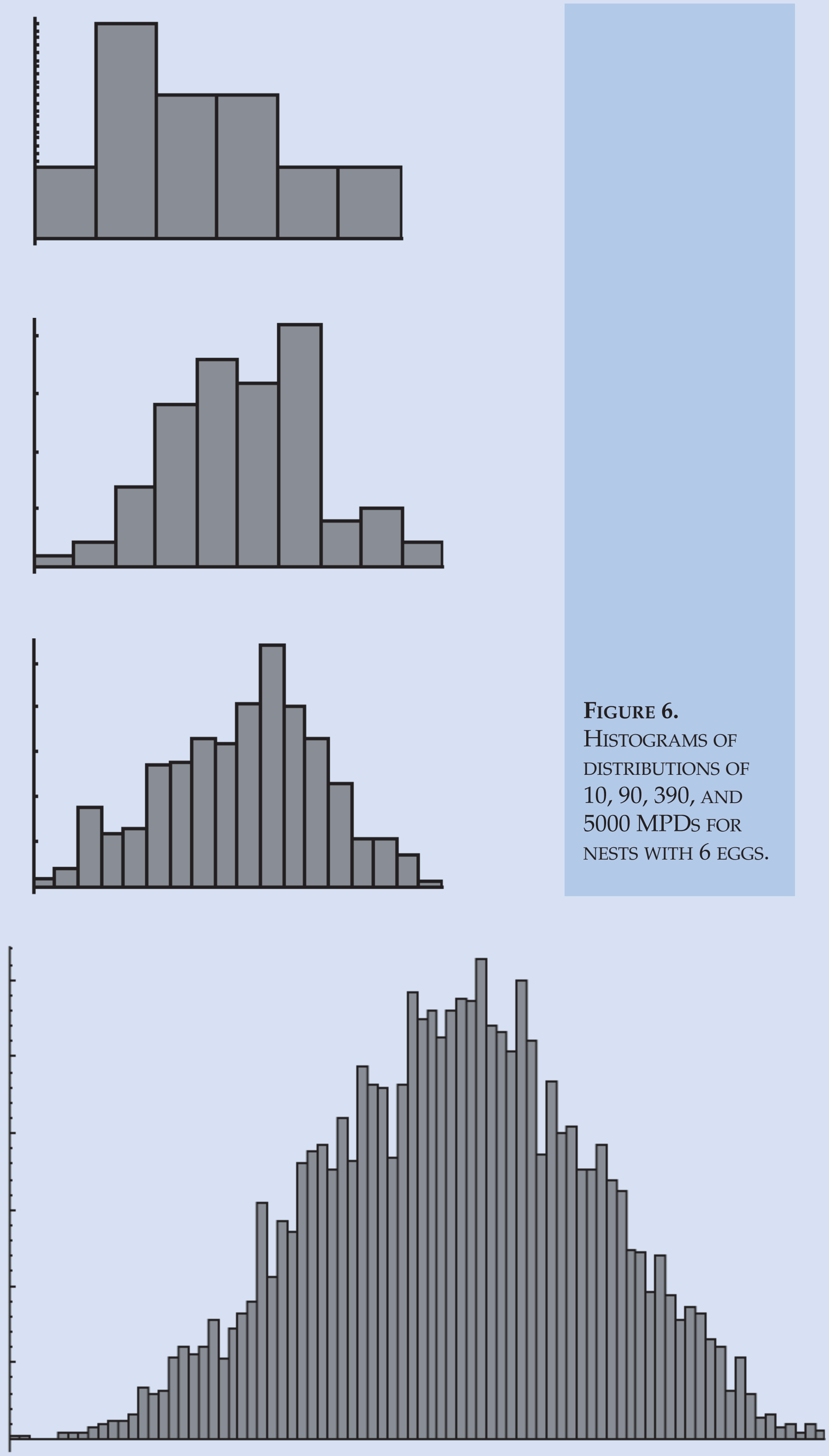
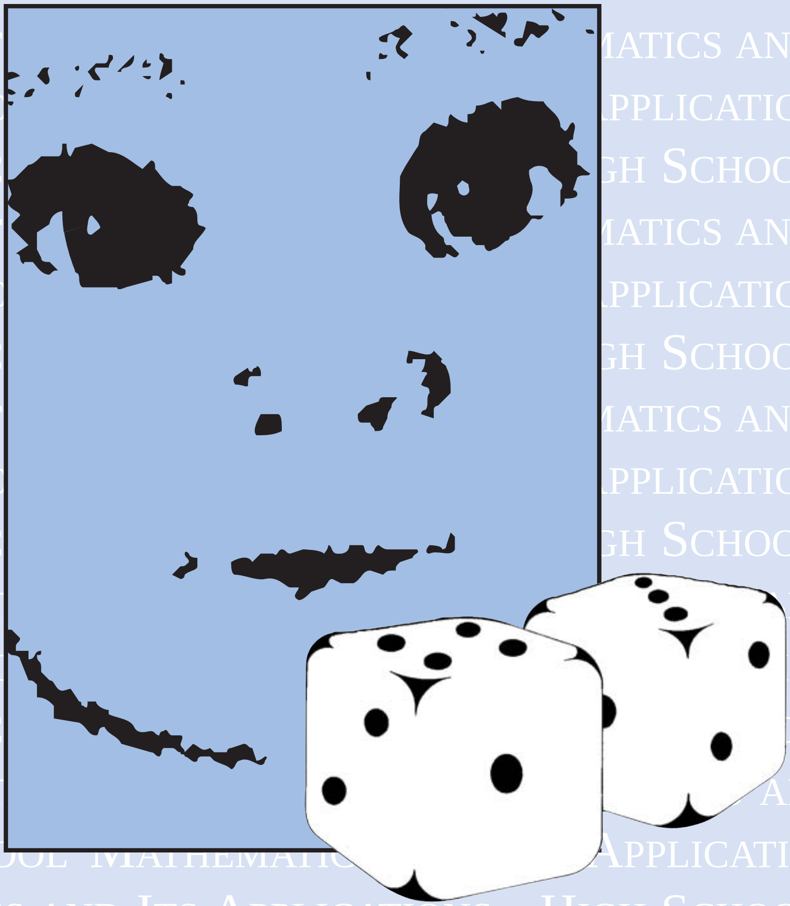


Figure 6. Histograms of distributions of 10, 90, 390, and 5000 MPDs for nests with 6 eggs.

CONSORTIUM 86

The HiMAP Pull-Out Section



Genetics and a Mathematically Indefensible Historical Movement

Rosalie A. Dance and
James T. Sandefur

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Mathematical modeling can help prevent harmful and expensive mistakes.

Mathematical modeling is an important tool in both governmental policy decision-making and in industrial planning. In mathematical modeling, we develop a function, a graph, an equation, or a simulation based on assumptions about a situation. The results often give insight into the situation. The expense of making a mathematical model is usually significantly less than making a prototype. Even more importantly, a math model can sometimes help us avoid making decisions that may have disastrous effects on humans and our environment.

In this article, we are going to develop and analyze some models related to population genetics. In particular, we are going to study how the genetic makeup of a population changes over time as a result of natural and man-made influences.

In 1927, the United States Supreme Court upheld a Virginia sterilization law, allowing the forced sterilization of a mentally retarded woman. It is estimated that by 1935, about 20,000 forced sterilizations had been performed in the United States alone.

We begin by developing models related to the failed "eugenics movement" of the late ninteenth and early twentieth century. This worldwide movement promoted forced sterilization of individuals deemed to have harmful genetic traits. The movement particularly targeted mental retardation, with the goal of eliminating mental retardation.

Pairs of genes determine many traits, one gene inherited from each parent. The genes come in different forms, called alleles. The particular pair of alleles inherited from the parents determines the trait exhibited by the child. For simplicity, we assume there are just two alleles for a certain gene; we will designate them A and B.

Sweden, in what is now seen as a national scandal, opened the "Swedish Institute for Racial Biology" in the 1920s. During the 1940s and 1950s, Sweden sterilized about 2000 people per year. Their program was not stopped until 1974. (Washington Post, 8/29/97)

The possible genotypes are AA, AB, BA, and BB, where the first allele is from the mother and the second allele is from the father. We will assume that A is a dominant allele and B is recessive, so that the AA, AB, and BA individuals exhibit the trait determined by the A-allele and the BB individuals exhibit the trait determined by the B-allele.

Let's assume that the fraction of alleles that are A and B among the parents is p and q, respectively. Since all alleles are of one type or the other, then $p + q = 1$. We assume that the genetic makeup of males and females are the same, so that the probability of a child getting an A-allele from either parent is p.

You Try It #1

Assume that $p = 2/3$ and $q = 1/3$. Simulate the births of a population of 36 children by rolling a pair of dice 36 times. Mark one die to represent the allele from the mother. The other die represents the allele from the father. If the numbers 1, 2, 3, or 4 come up, then that die corresponds to an A-allele being received from the corresponding parent. If a 5 or 6 shows, then a B-allele is received from that parent.

*A child with 2 A-alleles is called a **dominant homozygote.** A child with 2 B-alleles is called a **recessive homozygote.** A child with one allele of each type is called a **heterozygote.***

a) What fraction of the children in your simulation have two A-alleles? What fraction have one allele of each type? What fraction of the children have two B-alleles?

b) What fraction of the 72 alleles are A-alleles?

One approach to mathematical modeling is to simulate the situation. YTI#1 is a simulation that gives us some idea of what the genetic makeup of the next generation would look like. Normally, the simulation would generate a population larger than 36 to get a better sense of what is happening.

A deterministic model is one that predicts what will happen. In this genetics model, we compute the probability of a child being a dominant homozygote,

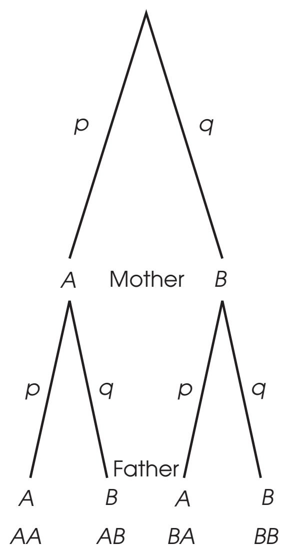
Prob(AA) = p^2.

Similarly, we get

Prob(AB or BA) = $pq + qp = 2pq$ and Prob(BB) = q^2 .

See the tree diagram for details.

Tree diagram for genetics



Multiply probabilities on branches to get results at end of branches.

You Try It #2

Suppose that $p = 2/3$ and $q = 1/3$.

a) What is the probability of a child having two A-alleles? One allele of each type? Two B-alleles?

b) Suppose 36 children are born. From the probabilities in part a), how many children do you expect to have AA, AB, BA, BB?

c) Using your answers to part b), what fraction of the 72 alleles do you expect to be A?

d) Compare your results to YTI#1.

With an elementary understanding of genetics, we can investigate what to expect if individuals with a recessive genetic trait are sterilized. In the model, we remove these individuals from the genetic pool for the next generation.

We would have predicted 89 AA children, 89 AB or BA children and, 22 BB children rounded to nearest integer.

You Try It #3

a) Suppose the BB children from YTI#1 were removed. What fraction of the alleles among the remaining children is B?

b) Suppose the BB-children are removed from consideration in YTI#2b. What fraction of the remaining alleles is B?

TI-83 Program for simulating a population

When you run the program, input the fraction of A-alleles for P (0.66667 in YTI#4). Then input the number of children you wish to generate (200 in YTI#4).

```
:ClrList L1
:Disp "P"
:Input P
:Disp "NUM CHILD"
:Input I
:{0,0,0}→L1
:For(J,1,I)
:1→N
:If rand<P
:N+1→N
:If rand<P
:N+1→N
:L1(N)+1→L1(N)
:End
:Disp "BB AB AA"
:Disp L1
```

We simulated births of 200 children using a programmable calculator. Instead of rolling a die, we generated 2 random numbers between 0 and 1 using a calculator's random number generator. When a number between 0 and 0.66667 was generated, it was interpreted as the child getting an A-allele from a parent. A randomly obtained number between 0.66667 and 1 modeled the child receiving a B-allele from a parent. The choice was made twice for each child, to model receiving an allele from each parent. The result was 96 AA children, 80 AB or BA children, and 24 BB children. The 48 B-alleles from the homozygous children were then removed from the gene pool. The remaining gene pool of this generation is 192 A-alleles from the 96 AA children, 80 A-alleles and 80 B-alleles from the heterozygote children, giving a total of 272 A-alleles and 80 B-alleles. Thus, for this generation,

$$p = \frac{272}{272+80} \approx 0.773 \text{ and } q \approx 0.227.$$

It is valuable to run a simulation several times to get a sense of (a) what results are typical and (b) how the results vary. We simulated a population of 200 children a total of 10 times and got the following results for q: 0.227, 0.225, 0.250, 0.219, 0.236, 0.251, 0.254, 0.250, 0.236, and 0.240. The average of these results is 0.24. This is our estimate for the proportion of *B*-alleles among the children's generation. We now denote $q_0 \approx 0.33$ and $q_1 \approx 0.24$ as the approximations for the genetic makeup of the initial generation and their children.

You Try It #4

a) Simulate the grandchildren's generation by generating populations of size 200 ten times, using $p = 0.76$. Find the fraction of alleles that are *B* among the *AA, AB,* and *BA* children. Approximate q_2, the proportion of *B*-alleles in the grandchildren's generation, by averaging your 10 results.

b) Use your result from part a) to generate 10 populations of size 200 among the great grandchildren's generation. Use the result to approximate q_3.

You Try It #5

Figure out how our calculator program works.

It is doubtful in many cases that people who were sterilized as a result of the eugenics movement had a genetic defect causing mental retardation. In fact, some of the people being sterilized may not have been mentally retarded.

a) What do the first 5 lines in the program do?

"{0,0,0}→L_1" establishes a list with three items in it, each a zero. The first value will tell us the number of *BB* births (so far), the second value tells us the number of *AB* or *BA* births, and the third value tells us the number of *AA* births.

"For (J,1,I)…End" is a loop that will run I times. J is the counter and begins with a value of 1. Each time it reaches the End statement, J increments by 1 until it reaches a value of I. Then it goes on to the next statement.

b) What happens inside the loop? First: what does the statement "1→N" do?

Suppose a recessive allele causes mental retardation and that this allele is relatively rare in the population; that is, B is small. The results of our computations indicate that it will take sterilizing many generations to have an appreciable affect on the prevalence of B in the population. Remembering that a generation is about 15 years, this indicates that eugenics is somewhat ineffective.

c) Second: What does the pair of lines "If rand<P (then) N+1→N" do? (*rand* is the random number generator; it produces a number at random between 0 and 1.) Why is the pair of lines "If rand<P (then) N+1→N" repeated?

d) In the last line of the loop, L_1(N) means the Nth item in our list, L_1. What does the statement "L_1(N)+1→L_1(N)" do? And, what does it mean in the context of our simulation?

e) After the loop runs 200 times (which it will do if we input a value of 200 for I), what will the sum of the numbers in the list L_1 be? What will each of the individual numbers in L_1 mean?

In the last 2 lines of the program, we output the results, with labels.

We are now going to predict the fraction of *A* and *B* alleles in each generation, assuming that the *B*-allele is a recessive trait and that all individuals exhibiting the trait caused by the *B*-allele are sterilized.

You Try It #6

A question that points up another problem with the eugenics movement is, "Who determines what traits are negative?"

a) Suppose $p_0 = 2/3$ and there are 200 children born to this generation. How many *AA, AB* or *BA,* and *BB* children do you expect? (Be exact; include the fraction.) How many *A* and *B* alleles do you expect among the *AA, AB,* and

BA children? Use these numbers to predict p_1 and q_1, *given that BB children do not reproduce.*

b) Repeat part a) using your calculation for p_1 and q_1 to predict p_2 and q_2.

c) Repeat part a) using your calculation for p_2 and q_2 to predict p_3 and q_3.

d) Assume that $q_n = 1/(n + 2)$. Use this to find p_n. Assume there are 200 children born to this generation. How many *AA*, *AB* or *BA*, and *BB* children do you expect? How many *A* and *B* alleles do you expect among the *AA*, *AB*, and *BA* children? Use these answers to predict q_{n+1}.

e) Suppose that $q_n = 0.04$. How many generations will it take until the proportion of *B*-alleles is reduced to 0.02?

f) How many generations will it take for the proportion of *B*-alleles to be reduced from 0.02 to 0.01?

g) How do the predicted results of parts a), b), and c) compare with the results of the simulations in YTI#4?

"Common sense" would seem to indicate that sterilizing individuals with harmful traits would reduce the trait in society. But the deterministic model and the simulations both cast doubt on the effectiveness of eugenics in reducing a "negative" trait from a small fraction of the population to an appreciably smaller one. These genetic models showing the ineffectiveness of eugenics were not difficult to develop. And yet this movement continued worldwide for decades.

Galactosemia is a genetic disease that prevents infants from metabolizing lactose and galactose. Galactose builds up in the blood, resulting in liver failure. Today, galactosemia is relatively easy to diagnose and treat.

We now turn our attention to a mathematical model that allows us to estimate a value that would be difficult to measure directly: mutation rates. We consider a situation in which the recessive trait caused by the *B*-allele is lethal, where all *BB* children die before reaching reproductive age. Historically, galactosemia was such a trait. Modeling this trait is analogous to modeling eugenics, where the *BB* adults were not allowed to reproduce. We add the additional assumption that a certain percent of the *A*-alleles mutate to *B*-alleles, as occurs with the allele for galactosemia.

The prevalence of the recessive allele for galactosemia is $q \approx 0.006$, so $p \approx 0.994$. This means that $q^2 \approx 0.00036$ of children are born with galactosemia. This is about 1 in 30,000.

Let's assume that $p_0 = 0.3$ and $q_0 = 0.7$ and that 16% of the *A*-alleles mutate to *B*-alleles. Using the same program as earlier to simulate a population of 200 children, we obtained 25 *AA* children, 88 *AB* or *BA* children, and 87 *BB* children. The genetic makeup of this generation is 50 *A*-alleles from the 25 *AA* children, 88 *A*-alleles and 88 *B*-alleles from the heterozygote children. None of the *BB* children survive. This gives a total of 113 *A*-alleles and 88 *B*-alleles.

Normal alleles mutate to the allele that causes galactosemia.

We now consider the mutation rate. Since 16% of the *A*-alleles, $(0.16)113 \approx 18$, mutate to *B*-alleles, we then have $113 - 18 = 95$ *A*-alleles and $88 + 18 = 106$ *B*-alleles. Thus, for this generation,

$$p_1 = \frac{95}{95+106} \approx 0.473 \text{ and } q_1 \approx 0.527.$$

You Try It #7

a) Simulate 200 children being born with $p_0 = 0.3$. Find the number of *AA* children and the number of *AB* or *BA* children. Use that to get the number of *A*-alleles and *B*-alleles among the children. Subtract 16% of the *A*-alleles and add that number to the number of *B*-alleles. Use the totals to estimate p_1 and q_1. Remember that no *BB* children will live to reproduce.

If a population reaches a point at which its genetic makeup remains about the same from one generation to the next, then the population is said to be in ***equilibrium.*** *This is what has happened in YTI#7.*

b) Using your estimates for p_1 and q_1, repeat part a) to get an estimate for p_2 and q_2.

c) Keep using your previous estimates for p_n and q_n to estimate p_{n+1} and q_{n+1} as you did in part b) until you have estimates for p_3 and q_3 through p_8 and q_8.

In YTI#7, you found that the fraction of alleles that are *A* and *B* seem to stabilize around positive values instead of one or the other going to 0. A deterministic model will help us understand what is happening:

You Try It #8

a) Suppose $p_0 = 0.3$ and that 200 children are born. Find the number of *AA* children and the number of *AB* or *BA* children expected. Use that to get the number of *A*-alleles and *B*-alleles among the children. Subtract 16% of the *A*-alleles and add that number to the number of *B*-alleles. Use the totals to estimate p_1 and q_1.

Notice that the proportion of A-alleles increases.

b) Suppose $p_0 = 0.8$ and 200 children are born. Find the number of *AA* children and the number of *AB* or *BA* children expected. Use that to get the number of *A*-alleles and *B*-alleles among the children. Subtract 16% of the *A*-alleles and add that number to the number of *B*-alleles. Use the totals to estimate p_1 and q_1.

Notice that the proportion of A-alleles decreases.

c) Suppose $p_0 = 0.6$ and 200 children are born. Find the number of *AA* children and the number of *AB* or *BA* children expected. Use that to get the number of *A*-alleles and *B*-alleles among the children. Subtract 16% of the *A*-alleles and add that number to the number of *B*-alleles. Use the totals to estimate p_1 and q_1.

Notice that the proportion of A-alleles remains constant, or is in ***equilibrium.***

In YTI#8, we saw that there was a value for *p* and *q* that remained constant from one generation to another. This value is the equilibrium genetic makeup for the population. We also saw from parts a) and b) that if the genetic makeup is not in equilibrium, then the genetic makeup of the next generation will be closer to the equilibrium value. This shows that over time the genetic makeup of this population stabilizes at equilibrium. Let's see why this happens.

Suppose 200 children are born to a population where *p* is the proportion of *A*-alleles and $q = 1 - p$ is the proportion of *B*-alleles. Then we expect $200p^2$ *AA* children and $400p(1 - p)$ *AB* and *BA* children. This gives a total of

$400p^2 + 400p(1 - p) = 400p$ *A*-alleles,

$400p(1 - p)$ *B*-alleles, and

$400p + 400p(1 - p) = 400p(2 - p)$ alleles in total.

Assume that the fraction of *A*-alleles that mutate to *B*-alleles is *x*. Then the number of *A*-alleles remaining after mutation is

In YTI#7 and YTI#8, x = 0.16.

$400(1 - x)p$.

Thus, we have

$$p_1 = \frac{400(1-x)p}{400p(2-p)} = \frac{1-x}{2-p}.$$

If x = 0.16, this equation becomes

$$p_n = \frac{0.84}{2 - p_{n-1}}$$

Check to see if it gives the same answers to YTI#8, parts a, b, and c with $p_{n-1} = 0.3, 0.8, 0.6$, respectively.

If x is known, then the equation

$$p_n = \frac{1-x}{2-p_{n-1}}$$

can be used to approximate p_n over time. Furthermore, at equilibrium $p_n = p_{n-1} = p$, so we solve the equation

$$p = \frac{1-x}{2-p}$$

to find that the equilibrium proportion of A-alleles is

$$p = 1 \pm \sqrt{x}.$$

We have discovered that the mutation rate for galactosemia is about 0.000036 or about one allele in 30,000.

Since $q = 1 - p$, if $p = 1 - \sqrt{x}$ then $q = \sqrt{x}$. This means that the fraction of children born with BB is $q^2 = x$. Thus, the fraction of children with the disease caused by the B-allele equals the mutation rate.

1 a You will probably get about 16 *AA* children, 16 *BA* or *AB* children, and 4 *BB*.

b About 2/3.

2 a Prob(*AA*) = 4/9, Prob(*AB* or *BA*) = 4/9, and Prob(*BB*) = 1/9.

b Predicts 16 are *AA*, 16 are *AB* or *BA*, and 4 are *BB*.

c 2/3

d It is likely that the results will be similar, but not the same.

3 a About $\frac{1}{4}$

b $\frac{1}{4}$

4 a You should get $q_2 \approx 0.20 = 1/5$.

b $q_3 \approx 0.17 \approx 1/6$.

5 a Clear List L_1 of any values it may have had from previous work. Ask for and receive input for a value to assign to P, the probability of an *A*-allele. Ask for and receive input for a value of I, the number of child births in the simulation you are going to do.

b "1→N" makes N = 1.

c This command chooses a random number and tests to see if it is less than the value of P. If it is less, this is interpreted as receiving an *A*- allele from the first parent. The command is repeated to determine whether an *A*-allele is received from the second parent.

d "L_1(N)+1→L_1(N)" adds 1 to one of the elements of the list. If both random numbers were ≥ P (of which there is probability 1 – P), we interpret this as the birth of a *BB* child. In this case, the value of N will not change; N will still be 1. So the statement causes "L_1(1)+1→L_1(1); that is, the first item in the list increases by 1. This means we have added 1 to the number of *BB* births. If one of the random number choices is < P and the other is ≥ P, we interpret this as an *AB* or *BA* birth and increase the value of *N* by 1; L_1(2)+1→L_1(2). Finally, if both random number choices are less than P, the "child" is *AA*, so the third number in the list increases by 1.

e The sum L_1(1)+L_1(2)+L_1(3) will be 200, or whatever number you input for I, the number of births you want to simulate. The list started at 0, 0, 0, and one of the entries increased by 1 each time the program went through the loop. It looped I times, so if we input 200 as the value of I, the sum of the outcomes will be 200. The individual entries give the number of *BB* births, the number of *BA* or *AB* births, and the number of *AA* births, in that order.

6 a $88\frac{8}{9}$ *AA* children, $88\frac{8}{9}$ *AB* or *BA* children, and $22\frac{2}{9}$ *BB* children. This gives $266\frac{2}{3}$ *A*-alleles and $88\frac{8}{9}$ *B*-alleles.

$q_1 = \dfrac{88\frac{8}{9}}{355\frac{5}{9}} = 1/4$ so $p_1 = 3/4$.

b 112.5 *AA* children, 75 *AB* or *BA* children, and 12.5 *BB* children. This gives 300 *A*-alleles and 75 *B*-alleles. $q_2 = \frac{75}{375} = 1/5 = 0.2$ so $p_2 = 4/5 = 0.8$.

c 128 *AA* children, 64 *AB* or *BA* children, and 8 *BB* children. This gives 320 *A*-alleles and 64 *B*-alleles. $q_3 = \frac{64}{384} = 1/6 \approx 0.167$ so $p_3 = 5/6 \approx 0.833$.

d $p_n = \frac{n+1}{n+2}$. There will be $200\left(\frac{n+1}{n+2}\right)^2$ *AA* children, $400\frac{n+1}{(n+2)^2}$ *AB* or *BA* children and $200\frac{1}{(n+2)^2}$ *BB* children. This gives $400\frac{n+1}{(n+2)^2}$ *B*-alleles and $400\frac{n^2+3n+2}{(n+2)^2}$ *A*-alleles. The number of *B*-alleles divided by the total number of alleles simplifies to $q_{n+1} = \frac{n+1}{n^2+4n+3} = \frac{n+1}{(n+1)(n+3)} = \frac{1}{n+3}$.

e If $q_n = 0.04 = 1/25$, then $n = 23$. We want $q_m = 1/50$, so $m = 48$. This is an additional 25 generations. If a generation is about 20 years, this is 500 years.

f If $q_n = 0.02 = 1/50$ then $n = 48$. We want $q_m = 1/100$, so $m = 98$. This is an additional 50 generations (another 1000 years).

g Results should be reasonably close.

7 a $p_1 \approx 0.5$ and $q_1 \approx 0.5$.

b $p_2 \approx 0.56$ and $q_2 \approx 0.44$.

c The results for p_j approach 0.6; q_j approaches 0.4.

8 a 18 *AA* children and 84 *AB* and *BA* children giving 120 *A*-alleles and 84 *B*-alleles. 19.2 *A*-alleles mutate to *B*-alleles giving 100.8 *A*-alleles and 103.2 *B*-alleles. Thus, $p_1 = 100.8/204 \approx 0.494$ and $q_1 \approx 0.506$.

b 128 *AA* children and 64 *AB* and *BA* children giving 320 *A*-alleles and 64 *B*-alleles. 51.2 *A*-alleles mutate to *B*-alleles giving 268.8 *A*-alleles and 115.2 *B*-alleles. Thus, $p_1 = 268.8/384 = 0.7$ and $q_1 = 0.3$.

c 72 *AA* children and 96 *AB* and *BA* children giving 240 *A*-alleles and 96 *B*-alleles. 38.4 *A*-alleles mutate to *B*-alleles giving 201.6 *A*-alleles and 134.4 *B*-alleles. Thus, $p_1 = 201.6/336 = 0.6$ and $q_1 = 0.4$.

HIGH SCHOOL MATHEMATICAL CONTEST IN MODELING OUTSTANDING PAPERS

HiMCM November 2003

The contest offers students the opportunity to compete in a team setting using applied mathematics in the solving of real-world problems.

Partial funding provided by IBM

Additional support provided by the National Council of Teachers of Mathematics (NCTM), the Mathematical Association of America (MAA), and the Institute for Operations Research and Management Sciences (INFORMS).

Editor's Comments

This is our sixth HiMCM Special Issue. Since space does not permit printing all of the nine national outstanding papers, this special section includes the summaries from six of the papers and edited versions of three. We emphasize that the selection of these two does not imply that they are superior to the other outstanding papers. They were chosen because they are representative and fairly short. They have received light editing, primarily for brevity. We also wish to emphasize that the papers were not written with publication in mind; the contest does not allow time to revise and polish. Given the 36-hour time limit, it is remarkable how well written many of the papers are.

We appreciate the outstanding work of students and advisors and the efforts of our contest director and judges. Their dedication and commitment have made HiMCM a big success. We also wish to note that this special section takes the place of our regular HiMCM Notes column, which will return in the next issue.

Contest Director's Article

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The High School Mathematical Contest in Modeling (HiMCM) completed its sixth year in excellent fashion. The growth of students, faculty advisors, and the contest judges is very evident in the professional submissions and work being done. The contest is still moving ahead, growing in a positive first derivative, and consistent with our positive experiences from previous HiMCM contests.

This year the contest consisted of 275 teams (a growth of about 30% from last year) from twenty-five states and from outside the USA. This year was the first year that schools were charged a registration fee of $45 for the first team and $25 for each additional team.

Thus our contest continues to attract an international audience. The teams accomplished the vision of our founders by providing *unique* and *creative mathematical* solutions to complex, open-ended, real-world problems. This year the students had a choice of two problems. Of the 275 teams, 156 submitted solutions to the B problem, and 119 submitted solutions to the A problem.

Problem A: What is it Worth?

In 1945, Noah Sentz died in a car accident and his estate was handled by the local courts. The state law stated that 1/3 of all assets and property go to the wife and 2/3 of all assets go to the children. There were four children. Over the next four years, three of the four children sold their shares of the assets back to the mother for a sum of $1300 each. The original total assets were mainly 75.43 acres of land. This week, the fourth child has sued the estate for his rightful inheritance from the original probate ruling. The judge has ruled in favor of the fourth son and has determined that he is rightfully due monetary compensation. The judge has picked your group as the jury to determine the amount of compensation.

Use the principles of mathematical modeling to build a model that enables you to determine the compensation. Additionally, prepare a short one-page summary letter to the court that explains your results. Assume the date is November 10, 2003.

Problem B: How Fair are Major League Baseball Parks to Players?

Consider the following major league baseball parks: Atlanta Braves, Colorado Rockies, New York Yankees, California Angles, Minnesota Twins, and Florida Marlins.

Each field is in a different location and has different dimensions. Are all these parks "fair"? Determine how fair or unfair is each park. Determine the optimal baseball "setting" for major league baseball.

	Outfield Dimensions					Wall Height			
Franchise	Left Field	Left Ctr	Ctr Field	Right Ctr	Right Field	Left Field	Ctr Field	Right Field	Area of Fair Ter
Angels	330	376	408	361	330	8	8	18	110,000
Braves	335	380	401	390	330	8	8	8	115,000
Rockies	347	390	415	375	350	8	8	14	117,000
Yankees	318	399	408	385	314	8	7	10	113,000
Twins	343	385	408	367	327	13	13	23	111,000
Marlins	330	385	404	385	345	8	8	8	115,000

COMMENDATION:

All students and their advisors are congratulated for their varied and creative mathematical efforts. The thirty-six continuous hours to work on the problem provided (in our opinion) a vast improvement in the quality of the papers. Teams are commended for the overall quality of work.

Again this year, many of the teams had female participation, which shows that this competition is for both male and female students. Teams again proved to the judges that they had "fun" with their chosen problems, demonstrating research initiative and creativity in their solutions.

JUDGING:

We ran three regional sites in December 2003. Each site judged papers for both problems A and B. The papers judged at each regional site were not from their respective region. Papers were judged as Outstanding, Meritorious, Honorable Mention, and Successful Participant. All regional finalist papers for the Regional Outstanding award were brought to the National Judging. For example, eight papers may be discussed at a Regional Final and only four selected as Regional Outstanding but all eight papers are brought and judged for the National Outstanding. The national judging chooses the "*best of the best*" as National Outstanding. The National Judges commend the regional judges for their efforts and found the results were very consistent. We feel that this regional structure provides a good prototype for the future of the contest's structure as it continues to grow.

JUDGING RESULTS:

National & Regional Combined Results

Problem	**National Outstanding***	Regional Outstanding	Meritorious	Honorable Mention	Successful Participant	Total
A	**4**	12	33	45	25	**119**
B	**5**	10	42	75	23	**155**
Total	**9**	14	51	104	33	**274**

GENERAL JUDGING COMMENTS:

The judges' commentaries provide comments on the solutions to each of the two problems. As contest director and head judge for the problems, I would like to speak generally about team solutions from a judge's point of view. Papers need to be very coherent, concise, and clear. Students need to restate the problem in their own words so that the judges can determine the focus of the paper. Papers that explain the development of the model, assumptions, and its solutions and then support the solution findings mathematically generally do quite well. Modeling assumptions need to be listed and justified but only those that come to bear on the team's solution (that can be part of simplifying the model). Laundry lists of assumptions that are never referred to in the context of the model development are not considered relevant and deter from the paper's quality. The model needs to be clearly developed, and all variables that are used need to be defined. Thinking outside of the "box" is also an ingredient considered important by judges. This varies from problem to problem but usually includes model extensions or sensitivity analysis of the solution to the teams' inputs. A clear conclusion and answers to specific scenario questions are all key components. The strengths and weakness section of the paper is where the team can reflect on the solution. Attention to detail and proofreading the paper prior to final submission are also important because the judges look for clarity and style.

CONTEST FACTS:

Facts from the 6th Annual Contest:

- A wide range of schools/teams competed including teams from Hong Kong.
- The 275 teams represented 60 institutions; 44 repeats and 16 new institutions.
- 44.36% of the teams had female participation. Forty-three of the 275 teams were all female.
- There were 953 student participants, 548 male (57.5%) and 395 female (42.5%).
- Schools from 25 states participated in this year's contest.

THE FUTURE:

This HiMCM contest that attempts to give the underrepresented an opportunity to compete and achieve success in mathematics endeavors appears well on its way in meeting this important goal.

We continue to strive to grow. Any school/team can enter the contest, as there will be no restrictions on the number of schools entering. A regional judging structure will be established based on the response of teams to compete in the contest.

Again, these are exciting times for our high school students. Mathematics continues to be more than learning skills and operations. Mathematics is a language that involves our daily lives. Applying the mathematical principles that one learns is key to future success. The ability to recognize problems, to formulate a mathematical model and solve it, to use technology, and to communicate and reflect on one's work is key to success. Students develop confidence by tackling ill-defined problems and working together to generate a solution. Applying mathematics is a team sport.

Advisors need only be a motivator and facilitator. They should permit students to be ***creative*** and ***imaginative***. It is not the technique that is fundamental, but the process that discovers how assumptions drive the techniques. Mathematical modeling is an art and a science. Through modeling, students learn to think critically; communicate efficiently; and be confident, competent problem solvers for the new century.

Contest Dates: Mark your calendars early: the next HiMCM will be held in November 2004. Registrations of teams are due in October 2004. Teams will have a consecutive 36-hour block within a window of about two weeks to complete their chosen problem. Teams can registrar via the worldwide web at www.comap.com.

HiMCM Judges Commentary

Problem A: What is it Worth?

Although some teams initially believed that this was a simple algebraic problem involving the time value on money (i.e., a Future Value = $P(1 + I)^R$ problem), it soon became apparent to the better teams that they needed to perform some modeling and critical analysis of the situation.

The judges were impressed with the creativity, quality of the analysis, and the writing by the teams modeling Problem A. Teams dealt with time value of money, and they dealt with the value of land over time. It was imperative for teams working with land value to assume or "pick" a location (county or state) in order to estimate through modeling the appreciation of the land's value. Doing this provided teams an opportunity to restate the problem in more manageable terms. This allowed successful teams to move from the general ill-defined problem to an attackable, more specific, defined problem.

Many teams attacked this problem only as a time value of money issue. A key aspect here was obtaining an interest rate or index growth rate over time. Research as to which values were the most appropriate was necessary. Many teams merely stated values. Judges expected the teams to develop a sub-model to determine this.

One of the items that distinguished the better papers was that those teams calculated the "worth" from various models (money, land, etc.) and then came to a final conclusion about compensation after considering all their possible outcomes.

Verification of models or model testing was also an important discriminator. Some teams tested their models to see if the results made common sense. Others compared their predictions with historical results that they were able to obtain via web sources. It is noted that some answers given by teams made little practical sense.

Problem B: How Fair are Major League Baseball Parks to Players?

This was the purest modeling problem of the two. This problem was ill defined because students needed to determine what *they* meant by "fair." Teams needed to step back and insure that they had a well-defined problem, defined "fair," and determined which aspects of "fair" they were trying to model.

Some teams did not define "fair" until after they completed their models, which was deemed too late by many judges.

The judges commented that the statement of assumptions with justification, style of presentation, and depth of analysis was very good. The better papers offered a good diversity of solutions. Problem B, in comparison to Problem A, appeared to lend itself better to consideration of a variety of assumptions and justifications.

It was critical for teams to define "fair" in order to compare the baseball parks. The judges were surprised that no team recommended changing the dimensions of ballparks in order to make them more "fair" based on their mathematical findings. The better papers considered such variables as altitude, wind, humidity, temperature, and other environmental factors in their model or their discussions of the model.

Many papers used computer code to determine the issue of fairness of the baseball parks. Computer codes used to implement mathematical expressions can be a good modeling tool. However, judges expect to see an algorithm or flow chart in the paper from which the code was developed. Successful teams provided some explanation or guide to their algorithm(s)—a step-by-step procedure for the judges to follow. The code that teams attached to their paper may only be read for those papers that reach the final rounds of judging. The results of any simulation or computer code need to be explained, and sensitivity analysis should be performed.

For example, consider an algorithm for the flip of a fair coin:

INPUT: Random number, number of trials

OUTPUT: Heads or tails

Step 1: Initialize all counters.

Step 2: Generate a random number between 0 and 1.

Step 3: Choose an interval for heads, like [0.0.5]. If the random number falls in this interval, the flip is a heads. Otherwise the flip is a tails.

Step 4: Record the result as a heads or a tails.

Step 5: Count the number of trials and increment: Count = Count + 1.

An algorithm such as this would be expected in the body of the paper with the code as an appendix.

The judges commend the teams for a truly outstanding job on a difficult, open-ended problem that provided some interesting reading.

GENERAL COMMENTS FROM JUDGES:

Executive Summaries:

These are still, for the most part, one of weakest parts of team submissions. These should be complete in ideas not details. They should include the "bottom-line" and the key ideas used. They should include the particular questions addressed and their answers. Teams should consider a three paragraph approach: restate the problems in their own words, give a short description of their method to model and solve the problem (without giving specific mathematical expressions), and state the conclusions, including the numerical answers in context.

Restatement of the Problem:

Problem restatements are important for teams to move from the general case to the specific case. They allow teams to refine many aspects of their thinking to give their model uniqueness and a creative touch.

Assumptions/Justifications:

Teams should list only those assumptions that are vital to the building and simplifying of their model. Assumptions should not be a reiteration of facts given in the problem description. Assumptions are variables (issues) acting or not acting on the problem. Every assumption should have a justification with it. Variables chosen need to be listed with notation and be well defined.

Model:

Teams need to show a clear link between the assumptions they listed and the building of their model or models.

Model Testing:

Model testing is not the same as testing arithmetic. Teams need to compare results or attempt to verify (even with common sense) their results.

Teams that use simulation must provide a clear step-by-step algorithm for the proposed simulation. Lots of runs and related analysis are required when using a simulation to model a problem. Sensitivity analysis is also expected to determine how sensitive the simulation is to the model's key parameters.

Conclusions:

This section deals with more than just results. Conclusions might also include speculations, extensions to the model, and generalizations of the model. Teams should ask themselves what other questions would be interesting if they had more time and then tell the judges about their ideas.

Strengths and Weaknesses:

Teams should be open and honest here. They should answer the question, "What could we have done better?"

References:

Teams may use references to assist in modeling the problem. However, they must also identify the sources. It is still required of the team to show how the model was built and why it was the model chosen for this problem. Teams are reminded that only inanimate resources may be used. Teams cannot call upon real estate agents, bankers, or any other real person to obtain any information related to the problem.

Adherence to Rules:

Teams are reminded that detailed rules and regulations are posted on the high school contests area of the COMAP Website. Teams are reminded that the 36-hour time limit is a consecutive 36 hours.

Problem A Summary: Clarkstown South High School

Advisor: Mary Gavioli

Team Members: Simi Bhat, Daniel Gendler, Mitchell Livingston, Terry Van Hise

When the rightful inheritance of a beneficiary is not given shortly after the probate court's original ruling, finding an appropriate amount of compensation for that inheritance is complicated by several factors. Noah Sentz's fourth child should have been awarded 1/4 of 2/3 of his estate according to state law at the time of Mr. Sentz's passing. However, the fourth child is now suing for compensation because he did not receive his portion of the estate. There were several factors we considered when calculating Mr. Sentz's compensation, the most important of which being inflation. Inflation is the progression of changing value of a dollar over time. This means that the items a dollar could purchase when Mr. Sentz passed away probably would not cover the price of that item at the present time. We found the inflation rate from the time that the three children sold their portions of the estate to the present to be 802.9%. This means that if the land was worth $1200 and other personal assets were valued at $100 then, then the land would be worth $9624 and the other personal assets would have a value of $802.90 now. We also took into account the revenue the fourth child would have been deprived of because he did not own the land since 1945. The total amount of revenue generated from the land would have been $53,568. However, the fourth son would have had to pay additional income tax on this revenue, which would amount to $13,894.80. He would have also had to pay property tax on this land for the past fifty-eight years, a total of $316.04. To find the final monetary compensation the son should be given we summed the values of all assets and income and subtracted all taxes giving Nick a final compensation of $49,784.06. We made a generalized model, allowing for the input of several variables such as state and federal income tax rate, original value of land, original value of personal assets, and such variables. To increase the ease of use of our model, we created a computer program in Java so that court officials could simply input the variables into the program and receive an output of the calculated monetary compensation the disputing beneficiary should receive. The math model we created uses data taken from government references and economic journals. We feel it is a feasible and accurate model.

Problem A Summary: Mills E. Godwin High School

Advisor: Ann Sebrell

Team Members: Derek Austin, Srini Sathyanarayanan, Matthew Walker, Zhiyuan Xu

We the jury have determined that the plaintiff, the fourth child of the late Noah Sentz, deserves monetary compensation for assets not awarded in the amount of $18,772.35. Through investigating the problem, we researched various aspects of inheritance laws as well as the economic factors influencing the asset price. Our amount of compensation is the modem value of his portion of the assets minus property taxes accrued since 1945. We found this amount of compensation through a mathematical model that we created.

We initially created a simple model in which the plaintiff would receive value only from land assets and built upon this. We established the property as rural farmland from Pennsylvania. We realized that rural farmland is highly influenced by inflation and the appreciation of land value. By tracking basic land value patterns, we were able to roughly estimate the value of land in 2003. Through researching Consumer Price Index we approximated the influence of the cost of living on the price of consumer goods, which we then included in the model as part of his compensation. We decided that the four children initially split the consumer goods while the mother received her portion of the estate only in farmland; in other words, we accounted for an uneven split in the type of assets distributed, even though the amount of assets for each group was even. Finally, we deducted a 9.5% property tax rate from the estimated value of the land portion of his share of assets to create a final amount of compensation.

Through thorough examination of various factors influencing the value of the estate, we can confidently conclude that the compensation for the fourth child's inheritance in this situation should be $18,772.35.

Problem B Summary: Maggie L. Walker Governor's School

Advisor: John Barnes

Team Members: Guilherme Cavalcanti, Thomas Fortuna, Mrinal Menon, Derek Miller

Our first step in evaluating whether or not the fields were fair was to hypothesize that teams with a greater penchant for hitting home runs would build smaller stadiums to hit more home runs, while teams that could not hit as many home runs would build larger stadiums in order to deprive other teams of home runs. We then gathered available team data for three to five years before each team built their current stadium, trying to determine if their home run performance in relation to their league affected the creation of their fields. No relationship was found between team performance and the pure dimensions of the field as far as home runs were concerned.

We then decided to test if the initial velocity of a baseball hit at each field was significantly different in a perfect, airless world. It was determined that there was no significant difference in initial velocity for each stadium. Research, however, revealed that there was a significant difference in the amount of home runs hit at each of the six assigned stadiums, specifically Coors Field. The only construction difference between Coors Field and all other stadiums was its extreme altitude. We then proceeded to modify our baseball projectile motion model to include air resistance, to account for changes in altitude and temperature. Two forms of Euler's method were used to model the trajectory of a baseball through a dense atmosphere. This updated model revealed that altitude and temperature were major factors in determining a field's fairness; hitting home runs at sea level fields required a much greater initial velocity.

We then proceeded to define fairness in our fields. Our first belief was that fields that are symmetrical about their centerlines (right and left field distance and wall heights are equal) are fair to both right-handed and left-handed hitters. Of the six stadiums, Pro Player Stadium of the Florida Marlins was the only one that could be considered fair. The remaining five teams in increasing fairness are the Denver Rockies, New York Yankees, Atlanta Braves, and Anaheim Angels. The Twins' stadium, being temperature controlled, did not apply to our model. We also believe that we can create a fair field distance by taking average distances of existing fields that are getting league average home runs, accounting for temperature and altitude variations. Using these two beliefs and our air resistance model, we can create a "fair" field knowing only the altitude, temperature, and uniform wall height.

Problem B Summary: Illinois Mathematics and Science Academy

Advisor: Steven Condie

Team Members: Jeffrey Chang, Alex Garivaltis, David Xu

To begin tackling the problem, we decided to use a national batting average of 0.25 as an indicator of fairness. Our vision of the "fairest" field was one that provided its players with a level of offensive and defensive opportunities consistent with the national average. Therefore, a ballpark that produced a batting average closest to the national average was considered to be the optimal setting. The model consisted of a computer program that would automatically calculate the players' positions on a Cartesian plane given the dimensions of a field. To determine the batting average for any of the given ballparks, we created an algorithm that would simulate 10,000 at-bats, taking into account appropriate ratios for strikeouts and foul-outs. The variables Θ (initial angle with respect to ground), v_0 (initial velocity of the ball), and Φ (angle of ball's direction of travel on Cartesian plane) were all randomized within reasonable ranges for each simulation.

The program then analyzed the trajectory of the ball, taking into account the wind speed and any effects from low air pressure. By calculating the position of the ball's landing spot, our model determined whether the ball would be caught (resulting in an out), would not be caught (resulting in a safe base advance), or would fly over the fence as a home run (counting as four base advances). Our simulation was able to compute the batting average by keeping track of the number of base advances over 10,000 simulations.

To assess the fairness of the stadiums played in by the Angels, Braves, Rockies, Yankees, Twins, and Marlins; we found the batting averages resulting from each ballpark's specific dimensions and environmental conditions. The resulting order of fairness for the ballparks, from fairest to least fair, was:

Yankees (0.287), Braves (0.294), Twins (0.308), Marlins (0.316), Angels (0.322), Rockies (0.341)

To determine the optimal setting in general we decided to employ an evolution-based simulation method. The program generated a random set of field dimensions within reasonable limits and calculated the expected batting average using 10,000 at-bat simulations. The computer then continued generating random ballparks, calculating the batting average every time. Whenever a ballpark resulted in a batting average that was closer to the national average than any previously checked ballpark, it became the new optimal setting. After repeating this process for a long time, the best setting naturally surfaced.

	Outfield Dimensions					Wall Height		
	Left Field	Left Ctr	Ctr Field	Right Ctr	Right Field	Left Field	Ctr Field	Right Field
Ideal Ballpark Dimensions (ft)	387	393	434	397	335	9.8	13.1	6.7

Problem B Summary: Dubuque Hempstead High School

Advisor: Karen Weires

Team Members: Tom Duggan, Josh Lichti, Cory McDermott, Brad Willenbring

To quantitatively compare the fairness of the given stadiums, we defined numerous variables and methods of measurement. A computer program was created to simulate the behavior of hits in each stadium, taking into account its dimensions and prevailing environmental conditions. The program also simulated the distance the outfielders need to move in order to field all non-homerun hits. This data was coupled with numerous other factors including backstop length and field orientation to determine each stadium's Defensive Bias Rating (DBR), a measure of how offensively or defensively biased the given ballpark is.

The DBR value was then factored into the calculation of another value, the Oppositional Equality Rating (OER), a measure of how well matched a given team is to their stadium. A team that hits well overall, stationed in a stadium that caters to a hitting team, is going to have a distinct advantage because they play 50% of their games on their home field.

The DBR and OER both proved to be very accurate models of real-life conditions among the stadiums in question. The Colorado stadium was determined to be the most unfair stadium based on its OER rating. Environmental conditions make hits in Colorado tend to fly further, coupled with a team that statistically (already) seems to prefer offense to defense. In other cases, such as Atlanta, the team's ability wasn't paired with its stadium, so the two canceled each other out in the OER rating.

Problem B Summary: Mills E. Godwin High School

Advisor: Ann Sebrell

Team Members: Deepa Iyer, Brandon Murrill, Omari Stephens, David Williamson

During our investigation of this situation, our team aimed to derive an optimal model for fairness in a baseball park. To achieve this goal, we created a list of ten factors that we felt could affect the fairness equilibrium of a field. We researched the specifications of these ten factors and found an abundance of data from various sources.

By analyzing the magnitude of the effect of each factor at all of the six parks, we created a score for each park that led us to our conclusions. We found that each park was biased toward the offense or the defense; none was completely fair.

We tested our models by comparing the classifications we generated to the historical conception of each park: whether the venue was considered more favorable for a hitter or a pitcher.

We then proceeded to develop an optimal, completely fair baseball environment. We strove to ensure that our park favored neither the offensive or defensive team. Finally, we generated two strategies to develop a fair stadium. The first method required different factors to favor either the offense or the defense; the total effect of all the factors remained at equilibrium. The second method called for nearly neutral values for every single factor, again yielding a neutral venue overall. We combined the two strategies to develop a realistic, optimal design for a truly fair field.

Problem A Paper: The Spence School

Advisor: Eric Zahler

Team Members: Jillian Bunting, Madeleine Douglas, Yi Zhou

PROBLEM RESTATEMENT

Noah Sentz died in a car accident in 1945. His wife received one-third of his estate, and the children received two-thirds. Over the next four years, three of the four children sold their shares back to their mother for $1300 each. Noah's assets were mostly comprised of 75.43 acres of land. In early November 2003, the fourth child sued for his rightful share. The judge ruled that he is due cash compensation.

LETTER TO THE COURT

We created a mathematical model for determining the plaintiff's compensation. We made several assumptions to produce an effective model. These included the dates of the death of Noah Sentz and the subsequent sale of his assets by his three sons, that the first son received only land, that the value per acre was uniform, that the non-land assets appreciated at the rate of inflation, and that the values of each child's share was equal in 1945.

Besides these, our model made no assumptions about the specific values of variables. Instead, all variables are available to be manipulated. This is a very effective way of determining compensation because it can be implemented in a variety of situations. This is especially important because certain factors (i.e., value per acre, appreciation of land) are specific to geographic regions.

The benefits of our model are that it gives a fairly good range of values for the compensation and that it leaves only one factor to be determined by the court (the value of land per acre in 1945). Even without determining the value per acre, our model tells us that the compensation due the fourth son is between $55,400.39 and $90,162.43. Our model is also strong because it made no arbitrary assumptions. The inflation rates between years were based on historical data. The land appreciation rate was inferred from national data.

We feel confident that our inflation and land appreciation rates are fairly accurate. Therefore, we respectfully recommend that the model be used to determine compensation after the price per acre is determined.

ASSUMPTIONS AND JUSTIFICATIONS

In 1945, the worth of each child's share was equitable. We assume this because none of the children sued within four years. The suit did not arise until 58 years later, implying that there was no problem for a long time.

The fourth son received non-land assets that did not appreciate as fast as the land. He is suing for the difference between what his share is worth now and one-sixth of what the estate would be worth today.

Mr. Sentz died in January of 1945. We assumed this in order to have a base date to calculate inflation.

The first, second, and third children sold their shares back to their mother in January 1947, January 1948, and January 1949, respectively. This allowed us to find inflation values. We chose dates spread evenly over the four-year period to reflect the changing inflations over those four years.

The value of the land per acre is uniform. This allowed us to relate our equations to one another, and simplifies the problem by eliminating a variable.

All other assets appreciate at the rate of inflation. This simplifies the problem.

The rate of land appreciation is constant and reflects the best-fit curve that we found (see **Figure 1** and **Appendix B).** This allowed us to create equations; with land appreciation as a variable, there is too much unknown. We considered it reasonable to assume that the land appreciation was roughly equal to the overall trend in the United States.

The first son only received land. We assumed this in order to simplify the problem and allow us to solve for the value of the land in 1945.

OUR APPROACH

Land appreciates faster than the rate of inflation (see **Appendix A** for a proof). Since other assets appreciate at the rate of inflation, their growth over a long period of time is less than that of land. As a result of a discrepancy in distributing land and other assets (the fourth child only received non-land assets), the fourth child's share has not appreciated at the same rate as the land. Thus, if we calculated how much a sixth of the entire estate would be worth now and how much the fourth child actually has, the difference would be how much the fourth child is entitled to.

VARIABLES

L, with subscripts corresponding to children and mother, is the amount of land in acres each received. A subscript of M refers to the mother, a subscript of 1 refers to the first child, a subscript of 2 refers to the second child, and so forth. V is the value per acre in 1945. A is the rate of appreciation of land per year since 1945. S, with subscripts corresponding to the subscripts of L, is the value of the non-land assets the children received in 1945. T is the total value of the estate in 1945. I is the inflation rate between 1945 and the given year. P is the total value of non-land assets in 1945. Finally, t is the number of years since 1945. From these variables, we generated these equations:

$P = S_2 + S_3 + S_4 + S_M$. The total value of the non-land assets is equal to the sum of what was distributed.

$T = 75.43V + P$. The total value of the estate is the sum of the value of the land and the value of the other assets.

$V(L_1 + L_2 + L_3 + L_M) = 75.43V$. The sum of all land distributed is equal to the total value of the land.

$L_1V = L_2V + S_2 = L_3V + S_3 = S_4 = (L_MV + S_M)/2$. This comes from the assumption that at the time of distribution, each child received a sixth of the estate, and the mother received a third.

At t years from January 1945, the value of person X's inheritance can be expressed: $(L_XV)(1 + A)^t + (S_X)(1 + I_{1945M})$.

The trend in a graph of national trends in land appreciation (Figure 1) appeared exponential. We ignored the dip at the end of the graph because the trend of exponential growth continues, as evidenced by **Figure 2.** We estimated coordinates on the land-value graph in Figure 1. From these we found an exponential model (see Appendix B). The general form of the exponential model was $y = ab^x$, where a is the starting value and b is the increase per year. We determined b was 1.0857. Thus land appreciates at a rate of 8.57% per year. Our calculated a was 100. Using this equation, we calculated the value of each child's share based on the value of the first brother's share, which was entirely land.

$1 + A = b = 1.0857$

$(L_1V)(l + A)^2 = 1300$

$(L_1V)(1.0857)^2 = 1300$

$(L_1V) = \$1102.87$, which is the value of each child's share in 1945.

To determine the value of the estate in 1945, we multiplied 1102.87 by 6 and obtained \$6617.21 = T. We looked up the inflation rate between January 1945 and November 2003: 940.45% (see Bibliography). So the fourth brother's share is now worth:

November 2003 Value = $1102.87(1 + I_{2003})$

$\$11474.81 = 1102.87(10.4045)$.

We were told that the estate was comprised mainly of land, which we took to mean that the land's value was more than 50% of the total estate value, but less than or equal to five-sixths of it, since the fourth brother received only non-land assets comprising one-sixth of the total. Thus, we created a range of compensation. If the land is half the total value, then its 1945 value is \$6617.21/2 = \$3308.605. Then the current value of the entire estate is:

$(1.0857)^{58}(3308.605) + 10.4045(3308.605) = \$424200.76$.

His portion would be a sixth of that, or \$70700.13. Then he should receive the difference between that figure and \$11474.81: \$59225.32. The value per acre would be: 75.43V = 3308.605, or \$43.86.

On the other hand, if land constituted everything but his portion, 5/6 of the total estate, the value of the whole is:

$(1.0857)^{58}(5514.34) + 10.4045(1102.87) = \$661101.91$.

Then his portion should be \$110183.65, which means he should receive \$98708.84. The value per acre is: 75.43V = 5514.34, or \$71.11 (see **Figure 3**). The range of compensation is then between \$59225.32 and \$98708.84. It should be closer to the higher value, as "mainly" seems to imply closer to five-sixths than to half.

The function that determines the compensation is (see **Appendix C**):

General: $$\frac{(1+A)^t(75.43V)+(1+I_{2003})(T-75.43V)}{6} - S_4(1 + I_{2003})$$

Specific: f(V) =
$$\frac{1.0857^{58}(75.43V)+10.4045(6617.21-75.43V)}{6} - 1102.87(10.4045).$$

STRENGTHS OF MODEL

1. The general model needs few assumptions because most of its components are variables. It is flexible in that if only a few values of variables are known, by manipulating the equations a solution can be determined. In this case, only land appreciation and rate of inflation, both of which were found online, were necessary to find a compensation range. Finding the best value in that range requires only the value of the land per acre.
2. The general model accounts for assets and property, as well as the percentage of the total each comprises.
3. The model is easy to use and is not caught up in accounting for thousands of possibilities that arise from ambiguity of the variables. To account for every possible range of values of each variable would spawn a convoluted model that may not yield an accurate answer.
4. The data needed to use the model are readily available.

5. The model is fairly comprehensive because it accounts for various factors, including land appreciation and inflation rate.
6. The model does not require a jury to use high-level math. Plugging in values and adding to get a total requires only sixth-grade skills.

WEAKNESSES OF THE MODEL

1. The assumption that the value of land was uniform was necessary for the general method. A difference in the value would make the problem more complicated, as it is nearly impossible to account for disparities without knowing other factors such as location.
2. The assumption that the assets appreciate at the inflation rate is unrealistic.
3. The assumption that the land appreciation trend is perfectly exponential ignores other factors that affect it, such as natural disasters and location. Also, the land appreciation was derived from one graph.
4. The assumptions of the date of Noah's death and the dates when the brothers sold their shares affect the rate of inflation and land appreciation. This would alter the total calculated value of the estate in 1945.
5. The assumptions that the first brother only inherited land and that he was the first to sell his shares are crucial. While the latter assumption is logical (the brother with the most land had to have sold his shares first or else his land would have appreciated and not have been worth the same as the assets of the other brothers), the former assumption was for convenience and affects the estate value in 1945.
6. The assumption that the inequity arose because of unequal land distribution is the basis for the entire model. However, it is not explicitly stated that such is the case.

APPENDIX A:

PROOF THAT THE RATE OF LAND APPRECIATION IS GREATER THAN THAT OF INFLATION

$$\frac{1}{6}\mathrm{T}(1 + \mathrm{I}_{2003}) < \frac{75.43V(1+A_{2003})^{58} + P(1+I_{2003})}{6}$$

We know that I_{2003} = 940.54% (or 9.4045).

$$\frac{1}{6}\mathrm{T}(1 + 9.4045) < \frac{75.43V(1+A_{2003})^{58} + P(1+9.4045)}{6}$$

$$10.4045\mathrm{T} < 75.43\mathrm{V}(1 + \mathrm{A}_{2003})^{58} + 10.4045\mathrm{P}$$

$$10.4045\mathrm{T} < 75.43\mathrm{V}(1 + \mathrm{A}_{2003})^{58} + 10.4045(\mathrm{T} - 75.43\mathrm{V})$$

$$10.4045\mathrm{T} < 75.43\mathrm{V}(1 + \mathrm{A}_{2003})^{58} + 10.4045\mathrm{T} - 784.81\mathrm{V}$$

$$784.81\mathrm{V} < 75.43\mathrm{V}(1 + \mathrm{A}_{2003})^{58}$$

$$10.4045 < (1 + \mathrm{A}_{2003})^{58}$$

Therefore, at t years from 1945, $(1 + \mathrm{A}_{1945+t})^t > 1 + \mathrm{I}_{1945+t}$. Thus land appreciation is always greater than the rate of inflation.

$$1 + \mathrm{A}_{2003} > 1.041$$

$$\mathrm{A}_{2003} > 0.041$$

APPENDIX B:

CALCULATION OF EXPONENTIAL FUNCTION OF LAND VALUES

Approximate points:

(0, 100) In 1945, the price of land was 100 billion dollars.

(32, 1000) In 1977, the price of land was 1000 billion dollars.

(35, 2000) In 1980, the price of land was 2000 billion dollars.

(38, 3000) In 1983, the price of land was 3000 billion dollars.

$f(0)/f(32) = (ab^0)/(ab^{32})$
$100/1000 = 1/b^{32}$
$100b^{32} = 1000$
$b^{32} = 10$
$b = 1.081$

$f(0)/f(35) = (ab^0)/(ab^{35})$
$100/2000 = 1/b^{35}$
$100b^{35} = 2000$
$b^{35} = 20$
$b = 1.089$

$f(0)/f(38) = (ab^0)/(ab^{38})$
$100/3000 = 1/b^{38}$
$100b^{38} = 3000$
$b^{38} = 30$
$b = 1.094$

We took the average of the three b-values to get 1.0857 = 1 + A.

APPENDIX C:

DERIVATION OF GENERAL FORMULA

$$\mathrm{T} = 75.43\mathrm{V} + \mathrm{P}$$

$$\mathrm{P} = \mathrm{S}_2 + \mathrm{S}_3 + \mathrm{S}_4 + \mathrm{S}_\mathrm{M} \text{ or } \mathrm{P} = \mathrm{T} - 75.43\mathrm{V}$$

$$\mathrm{T} = 75.43\mathrm{V} + \mathrm{S}_2 + \mathrm{S}_3 + \mathrm{S}_4 + \mathrm{S}_\mathrm{M}$$

Since land appreciates uniformly, and non-land assets appreciate at the rate of inflation, the value of the estate in November 2003 is:

$$75.43\mathrm{V}(1 + \mathrm{A})^{58} + (\mathrm{S}_2 + \mathrm{S}_3 + \mathrm{S}_4 + \mathrm{S}_\mathrm{M})(1 + \mathrm{I}_{2003})$$

$$75.43\mathrm{V}(1 + \mathrm{A})^{58} + (\mathrm{T} - 75.43\mathrm{V})(1 + \mathrm{I}_{2003})$$

One-sixth of the estate value today is:

$$\frac{75.43V(1+A)^{58} + (T - 75.43V)(1+I_{2003})}{6}$$

The value of the fourth brother's share today is $\mathrm{S}_4(1 + \mathrm{I}_{2003})$, which we know is less than $\frac{75.43V(1+A)^{58} + (T - 75.43V)(1+I_{2003})}{6}$.

To find his compensation, find the difference between these:

$$\frac{75.43V(1+A)^{58} + (T - 75.43V)(1+I_{2003})}{6} - \mathrm{S}_4(1 + \mathrm{I}_{2003}).$$

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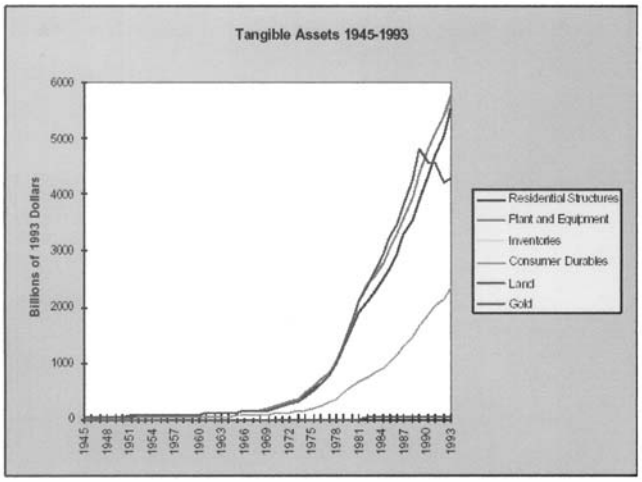

Figure 1. Land Appreciation Trend 1945–1993

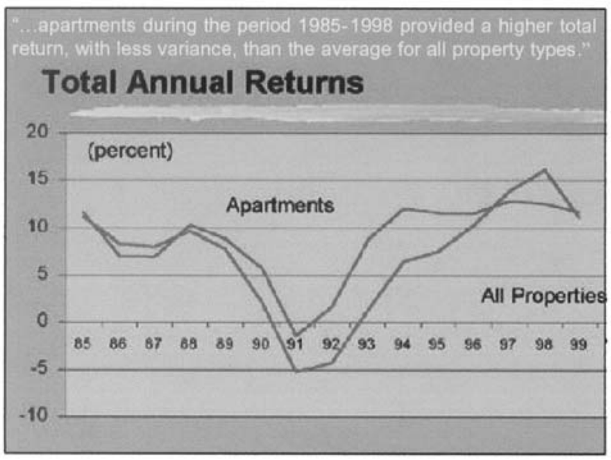

Figure 2. Total Annual Returns on Land 1985–1999

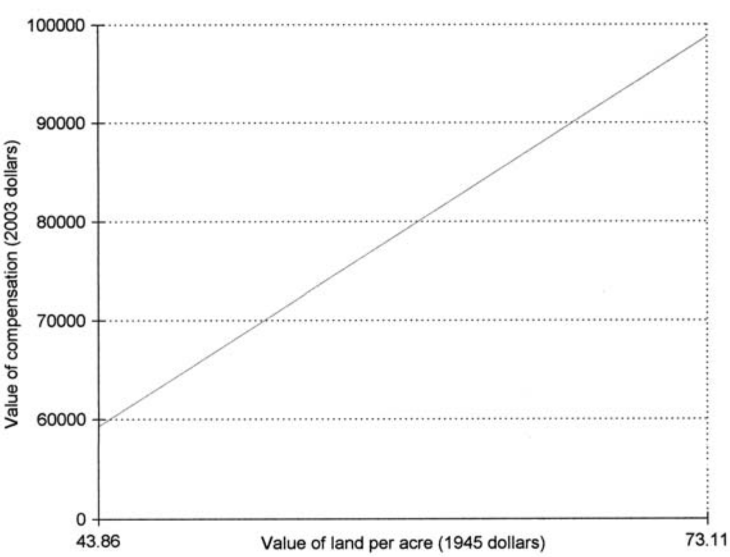

Figure 3. Compensation Range

Problem B Paper: Evanston Township High School

Advisor: Peter DeCraene

Team Members: Erica Cherry, Chris LeBailly, Eli Morris-Heft, Jean Rudnicki

OUR PARAMETERS FOR FAIRNESS

1. Left-handed batters should gain no advantage over right-handed batters. To ensure this, our field is symmetric about a line through home plate and second base.
2. Given a ball speed (V_{BALL}) and angle of elevation (φ), a ball should be a home run no matter the angle (θ) with respect to the foul lines at which it is hit.
3. The field should comply with major league rules and traditions, one of which is that the distance from home plate to the centerfield fence should be more than the corresponding distances down the foul lines. To adhere to this and comply with parameter 2, the fence is lowered in the middle such that a ball that would barely clear the right-field (or left-field) fence would, counting for distance, barely clear the centerfield fence. The fence is smoothly lowered from right field to center and then smoothly raised from center to left accordingly.
4. The current average number of home runs is about two per game. We aim to keep this figure static.
5. We recognize that, even as we strive to achieve parameter 1, there is no way to resolve the fact that a left-handed batter gets a one-step advance towards first base. We believe that the time it takes to step across home plate is negligible, so this discrepancy is not corrected.
6. We want the amount of fair territory to be consistent with the information given in the problem.

PROCEDURE

We constructed a computer simulation that picked a batter of random handedness, threw a random pitch, hit the ball at a random velocity and angle, and determined whether it was a home run. With each simulation of a game, we found the number of home runs. Through research, we found that the number of hits per game is about 20, and we worked with the dimensions of our park until, in accordance with fairness parameter 4, about two home runs were scored per game. We also ranked in fairness the parks given in the problem.

Our simulation variables are summarized in **Table 1** and **Figures 1 and 2**.

The parameter bounds were chosen based on our research. 10% of the general population is left-handed; we were unable to find the statistic for baseball players. The fastball is the pitch from which the most home runs are hit and is also the straightest pitch and the one that flies farthest when hit. The range of θ is derived naturally from the park's shape.

PROCEDURE

We researched equations to describe the flight of a hit baseball based on the speed of the pitch, the speed of the bat, and the initial angle of elevation, taking air friction into account. Our first equation shows how much kinetic energy is transferred to the ball:

$$KE_{BALL} = \frac{1}{3}\left(KE_{PITCH} + KE_{BAT}\right) \quad (1)$$

Only a third of the energy is transferred—the rest goes to vibration of the bat and to friction. Replacing KE with $\frac{1}{2}MV^2$, we have:

$$\frac{1}{2}M_{BALL}V_{BALL}^2 + \frac{1}{3}\left(\frac{1}{2}M_{BALL}V_{PITCH}^2 + \frac{1}{2}M_{BAT}V_{BAT}^2\right) \quad (2)$$

$$V_{BALL}^2 = \frac{1}{3}\left(V_{PITCH}^2 + \frac{M_{BAT}}{M_{BALL}}V_{BAT}^2\right) \quad (3)$$

$$V_{BALL} = \sqrt{\frac{1}{3}\left(V_{PITCH}^2 + \frac{M_{BAT}}{M_{BALL}}V_{BAT}^2\right)} \quad (4)$$

Thus we have an equation for the velocity of a hit ball.

Since we are designing the park for the real world, we have to consider air friction. Taking $\vec{v}$ to be the velocity vector of the ball (note that V_{BALL} is a scalar) and $\vec{f}$ as the drag vector, we have:

$$\vec{f} = -D\left|\vec{v}\right|\left(\vec{v}\right) \quad (5)$$

We have a negative on the right side because $\vec{f}$ is contrary to the motion of the ball. Breaking $\vec{f}$ into component vectors, we get:

$$\vec{f} = \left\langle \vec{f_x}, \vec{f_y} \right\rangle = \left\langle -D\left|\vec{v}\right|\left(\vec{v_x}\right), -D\left|\vec{v}\right|\left(\vec{v_y}\right) \right\rangle \quad (6)$$

Totaling the sum of the forces acting on the ball:

$$\sum F_x = -D\left|\vec{v}\right|\left(\vec{v_x}\right) = m\vec{a_x} \text{ and } \sum F_y = -mg - D\left|\vec{v}\right|\left(\vec{v_y}\right) = m\vec{a_y} \quad (7)$$

$$\vec{a_x} = \frac{D}{m}\left|\vec{v}\right|\left(\vec{v_x}\right) \text{ and } \vec{a_y} = -g - \frac{D}{m}\left|\vec{v}\right|\left(\vec{v_y}\right) \quad (8)$$

D is a constant dependent on air density (ρ), the ball's silhouette surface area (A), and the drag coefficient (C):

$$D = \frac{\rho CA}{2} \quad (9)$$

We researched ρ for each city in which we were given and for Evanston, where we located our ideal park. The silhouette surface area is easily determined because major league regulations state that the ball "shall…measure not less than nine nor more than $9\frac{1}{4}$ inches in circumference." Taking the circumference as $9\frac{1}{8}$ inches, A is $\frac{5329}{256\pi} \approx 6.626$ in^2. We modeled C on a graph in *The Physics of Baseball*. We tried many regression curves and found that the best fit was the cubic:

$$C(V_{BALL}) = (-7.773173 \times 10^{-5})V_{BALL}^3 + 0.013016 \times V_{BALL}^2 - 0.726969 \times V_{BALL} + 0.999441 \quad (10)$$

In order to derive a function for the position of the ball at time t, we split the forces acting on the ball into x- and y-components. To

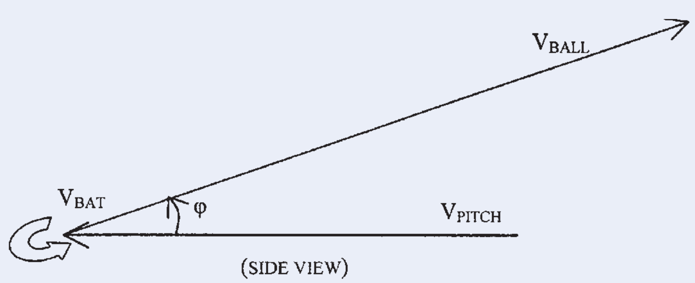

Figure 1.

Handedness of Batter	90% right; 10 % left
Velocity of Pitch (V_{PITCH})	90 ± 5 mph
Velocity of Bat (V_{BAT})	71 ± 2 mph
Initial Angle of Elevation of Hit Ball (φ)	35° ± 5°
Angle of Hit Ball with respect to line A (θ)	–45° to 45°

Table 1. Simulation variables

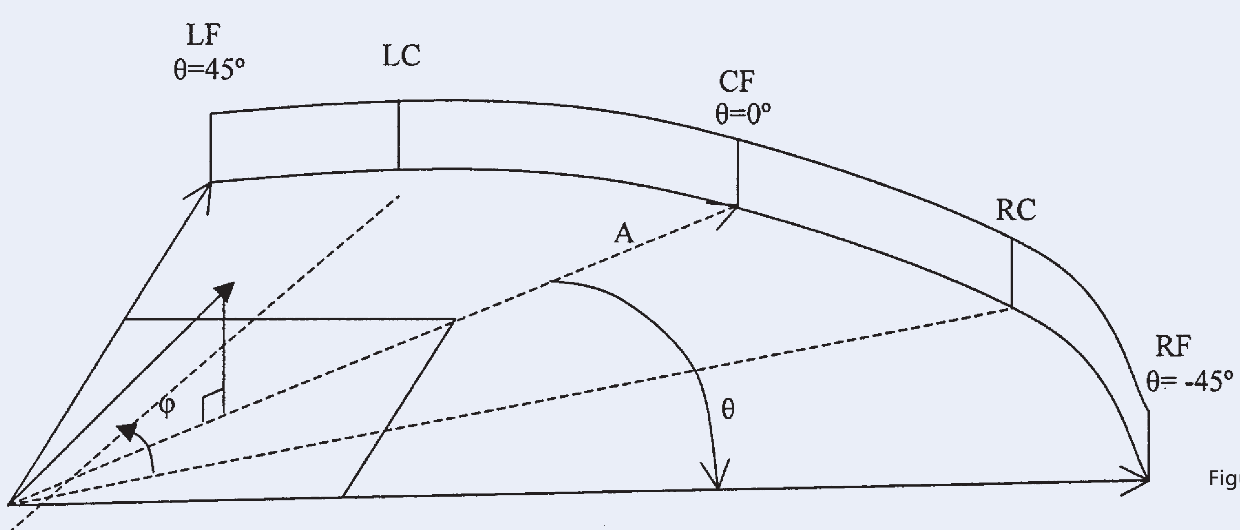

Figure 2.

derive the x-portion of the ball's position, we started with the equation:

$$F_x = ma_x \tag{11}$$

We also know that:

$$F_x = -kv_x^2 = ma_x = m\frac{dv_x}{dt} \tag{12}$$

because the only force acting on the ball in the x-direction is air friction. We are using k to represent the drag coefficient (about 0.0007). Using symbol manipulation, we find:

$$\frac{1}{v_x^2}dv_x = \frac{k}{m}dt \tag{13}$$

Integrating both sides, we get:

$$\int_{v_{0x}}^{v_x(t)} \frac{1}{v_x^2}dv_x = \int_0^t -\frac{k}{m}dt \tag{14}$$

$$\frac{1}{v_x(t)} - \frac{1}{v_{0x}} = \frac{k}{m}t \tag{15}$$

A little more manipulation gives us:

$$v_x(t) = \left(\frac{k}{m}t + \frac{1}{v_{0x}}\right)^{-1} \tag{16}$$

This gives us the velocity of the ball in the x-direction at time t. We also want the position at time t, which we derive by integrating:

$$p_x(t) = \int v_x(t)dt = \int\left(\frac{k}{m}t + \frac{1}{v_{0x}}\right)^{-1}dt = \frac{m}{k}\ln|kv_{0x}t + m| \tag{17}$$

In the y-direction, equations must take into account both air friction and gravity. We can start the same way as with the x-direction:

$$F_y = g - kv_y^2 = ma_y = m\frac{dv_y}{dt} \tag{18}$$

We can make this into a first-order differential equation by dividing m from the second and fourth terms in equation (18):

$$\frac{dv_y}{dt} = \frac{g - kv_y^2}{m} \tag{19}$$

Using equation (19), we formed a slope field and used the initial condition $v_y = 52.45 \sin(35°)$ to plot a solution to the differential equation. We took points from this curve and did a polynomial regression:

$$v_y(t) = -5.471729x^4 + 0.032844x^3 - 0.432209x^2 - 7.650204x + 52.326 \tag{20}$$

Thus, position function for the y-direction is:

$$v_y(t) = \int v_y dt \approx -1.094346 \times 10^{-4}t^5 + 0.008211t^4 - 0.14407t^3 - 3.825102t^2 + 52.326t + 1 \tag{21}$$

If we combine equations (17) and (21), we can find equations for the position and velocity of the ball at time t:

$$\vec{p}\left\langle \overrightarrow{p_x(t), p_y(t)} \right\rangle =$$

$$\left\langle \overrightarrow{\frac{m}{k}\ln|kv_{0x}(t) + m|, -1.094346\times10^{-4}t^5 + 0.008211t^4 - 0.14407t^3 - 3.825102t^2 + 52.326t + 1} \right\rangle \tag{22}$$

$$\vec{v}\left\langle \overrightarrow{v_x(t), v_y(t)} \right\rangle =$$

$$\left\langle \overrightarrow{\left(\frac{k}{m}t + \frac{1}{v_{0x}}\right)^{-1}, -5.471729x^4 + 0.032844x^3 - 0.432209x^2 - 7.650204x + 52.326} \right\rangle \tag{23}$$

We used a computer simulation to calculate the position of the ball. The simulation uses a fourth-order Runge-Kutta method. Using the changing acceleration and velocity vectors, it calculates each position of the ball based on the previous. As this method only works with first-order differential equations, the acceleration vector presents a problem. We resolved the problem by substituting in such a way that the acceleration vector's second-order equation became a system of first-order equations.

By using equation (8) to find the acceleration in both the x- and y-directions, it was possible to use Runge-Kutta to solve the system. Each time the Runge-Kutta algorithm was called, it moved the baseball forward one time step. Since there was a change in velocity in this step, it was necessary to recalculate the drag coefficient according to equation (10). In this simulation, the time step was 0.05 seconds. The Runge-Kutta algorithm was iterated until y-displacement was negative, meaning the ball had hit the ground.

When the Runge-Kutta method is called, the position and velocity vectors are stored in a matrix. We wanted to find the ball's height at a given horizontal displacement. To do this, the program searched the matrix and interpolated between the point with x-displacement just greater than what we wanted and the point with x-displacement just less than what we wanted.

This algorithm was very useful for figuring out if a hit was a home run. If we knew how far the ball had to travel to reach the wall, we used the previous algorithm to find the horizontal displacement. If the vertical distance was greater than the height of the wall, then the hit was a home run.

The next task was to find the distance from home plate to the wall. We knew the distance from home plate to left field, left centerfield, centerfield, right centerfield, and right field. Since we did not know what function modeled the shape of the wall, we assumed that it was linear. We broke it up into even sections, each with a range of 18° **(Figure 3)**.

Determining the value of θ depended on the batter's handedness. Right-handers tend to hit the ball towards left center, and left-handers tend to hit it towards right center. A Gaussian distribution was created to randomly pick the direction the ball was hit. For right-handers, the mean was θ = 22.5° (left centerfield) and for left-handers, the mean was θ = –22.5° (right centerfield). For both of these distributions, the standard deviation was set to 5°. A random number generator picked values according to the normal curve. These values were then altered to fit the appropriate distribution.

To find the initial speed of a hit ball, we had to know the speed at which it was pitched and how fast the batter swung. From these, we used equation (4) to determine the speed of the ball off the bat. We knew that the average fastball is thrown at about 90 miles per hour. From the Gaussian distribution (with a mean of 90 and a standard deviation of 5), we randomly selected the speed of the

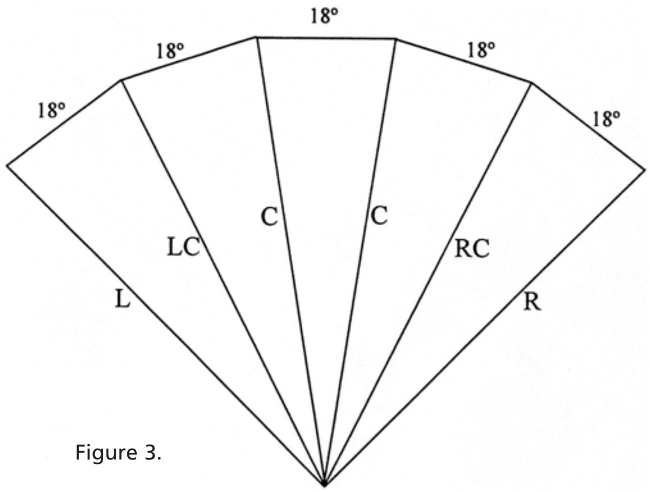

Figure 3.

pitch that fit the described distribution. Using the same method (with a mean of 71 and a standard deviation of 2), we found the bat speed. To calculate the angle of elevation (φ) we used the same method (with a mean of 35 and a standard deviation of 5). It was then possible to model the trajectory of the ball.

Given the distance from the fence to home plate, the simulation calculated the height of the ball and decided whether it was a home run, which only happens if the height of the ball is greater than the fence. However, the park specifications only include distances for left field, left centerfield, centerfield, right centerfield, and right field. We needed a way to transition smoothly from one given distance to the next. Though our park has a curved back wall, we assumed it was linear between points for simplicity.

In **Figure 4,** a and b are two of the six lengths given for each park. Let the third side of the triangle be c. The angle marked by the arrow is 22.5° because the fair territory is 90° and is divided into 5 sections (Figure 3). Let the distance to the fence be y (Figure 4). Let α be a varying angle, thus determining y. Note also angle β opposite side b. By the law of cosines:

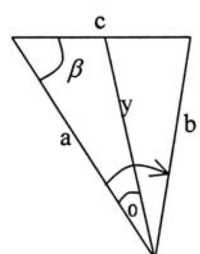

Figure 4.

$$c = \sqrt{a^2 + b^2 - 2ab\cos(22.5°)} \tag{24}$$

By the law of sines:

$$\beta = \arcsin\left(\frac{b\sin(22.5°)}{c}\right) \tag{25}$$

By the law of sines:

$$\frac{y}{\sin(\beta)} = \frac{a}{\sin(180° - \beta - \alpha)} \tag{26}$$

Solving for y gives us:

$$y = \frac{a\sin(\beta)}{\sin(180° - \beta - \alpha)} \tag{27}$$

We were able to use equations (24), (25), and (27) to find intermediate distances. These three equations were entered into the simulation to define where the wall was so the computer could designate whether a hit was a home run.

We then had to determine the height of the wall so that the probability of hitting a home run was independent of θ. After looking at various major league fields, we decided that the distance to the wall along each foul line should be 350 feet and that the distance to the centerfield wall should be 400 feet. We then needed an equation for the distance to the wall as a function of θ. We decided that the shape of the outfield wall should be an ellipse and used the polar equation of an ellipse:

$$r = \frac{2800}{\sqrt{30\sin^2\theta + 49}} \tag{28}$$

We needed to keep a few things constant. We wanted the wall height to be at least 8.5 feet at all angles to prevent outfielders from reaching over the wall. We set the ball speed and angle of elevation so that the ball just cleared the fence at all values of θ. To do this, we determined that the speed of the ball just after being hit should be 53.45 m/s and the angle of elevation should be 40°. After setting these constants, we ran the simulation.

According to our fairness parameters, a hit that just cleared the wall at one angle should just clear the wall at every other angle. Therefore, we used the simulation to determine the largest wall height that still allows a home run to clear the wall. We ran the simulation at 1° increments of θ between –45° and 45°. We then used the data and regression to find an equation for the wall height in meters in terms of θ:

$$Height(\theta) = 10.412518 \sin(3.304047\theta - 1.570796) + 13.082055 \tag{29}$$

In order to get the area of fair territory, we took an integral of the ellipse:

$$\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{4}} Height(\theta)^2 d\theta = 113873.328773 \text{ square feet} \tag{30}$$

The results in **Tables 2 and 3** show that the park we designed was the fairest. The difference between the percentages of right- and left-handed home runs was the smallest. It is also a symmetric field about the line from home plate to centerfield. In accordance with our second parameter, the probability of hitting a home run is indeed independent of θ because of the varying wall height. Any ball that is a home run at one angle would be a home run at any other angle. Originally, we wanted our average home runs per game to be about 2. In our simulation, it was closer to 1. However, our simulation assumes that all pitches are fastballs, so our estimate for the number of hits per game is probably low. Thus, we feel that we have achieved that parameter. Our last parameter was that our park should have an area of fair territory that is comparable to other parks. Our park is in the median range of the parks given in the problem.

The Braves' park had the second-smallest discrepancy between the right- and left-handers. However, this park had the third highest number of home runs per game, so it is an easy park whether the batter is left-handed or right-handed.

The Marlins' park had a higher discrepancy between left- and right-handers, but at least their number of home runs per game

Team	% of hits that are home runs	Number of home runs per game	% right-handed hits that are home runs	% left-handed hits that are home runs
Angels	39.97	7.994	35.63	78.88
Braves	27.88	5.576	26.69	39.59
Rockies	63.65	12.73	60.17	95.09
Yankees	21.05	4.21	19.34	37.42
Twins	23.99	4.798	18.82	70.64
Marlins	20.82	4.164	19.05	35.69
Evanston	5.53	1.106	5.51	5.69

Table 2. Data for First Simulation Run

Team	% of hits that are home runs	Number of home runs per game	% right-handed hits that are home runs	% left-handed hits that are home runs
Angels	39.7	7.94	35.29	79.66
Braves	27.39	5.478	25.86	40.74
Rockies	62.80	12.56	59.13	96.91
Yankees	21.24	4.248	19.54	37.21
Twins	24.23	4.846	19.47	68.06
Marlins	21.43	4.286	19.70	36.76
Evanston	5.41	1.082	5.37	5.75

Table 3. Data for Second Simulation Run

was closer to the league average of 2. The Yankees' park's fairness was similar to the Marlins'.

The Rockies' park had the third highest advantage for left-handers. However, the percentage of left-handed hits that were home runs was 95.09% and 96.91%; it should not be that easy for batters to hit home runs. In addition, the number of home runs per game was significantly above the league average of 2. Therefore, this park was one of the least fair.

The Twins' park and then the Angels' had the largest advantage for left-handers, making both very unfair. The Angels' park had more home runs per game, so it was slightly less fair than the Twins', but both were significantly less fair than Evanston.

Figure 5 is a diagram of our ideal park.

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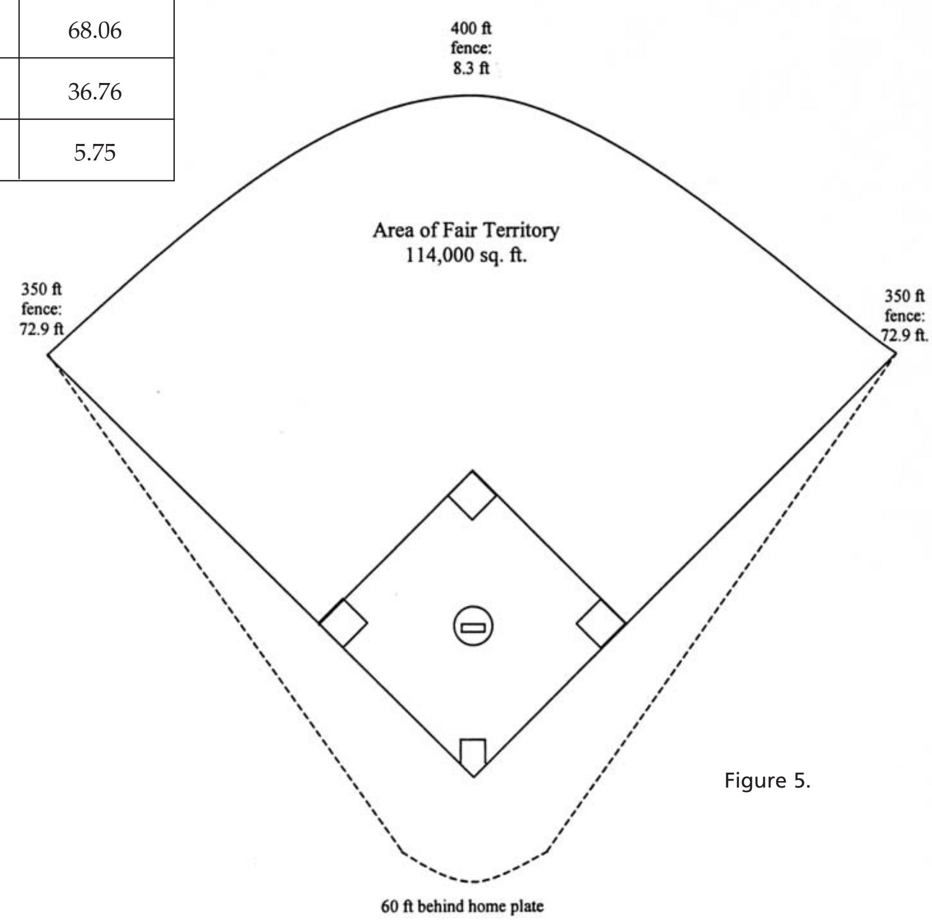

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Problem A Paper: Arkansas School for Mathematics and Sciences

Advisor: Bruce Turkal

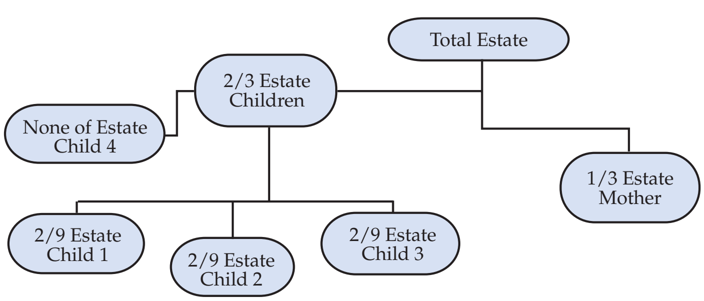
Team Members: Katherine Herring, Audrey Morris, Alex Wong, Johnson Wong

RESTATEMENT OF THE PROBLEM:

In 1945, the death of Noah Sentz resulted in the division of his estate among his wife and four children. According to state law, one-third of his property and assets went to his spouse and two-thirds went to his children. His estate mostly consisted of 75.43 acres of land. From 1946 to 1949, three of the children sold their shares back to their mother for $1,300 each. In the process of distributing the assets, his fourth child was left out under some unknown circumstances. This week the fourth child filed a lawsuit against the estate for his original inheritance from the probate case. The judge has ruled that the son shall receive his inheritance in the form of monetary payment. Our objective is to decide how much the fourth child will rightfully receive.

ASSUMPTIONS AND JUSTIFICATIONS:

- The problem occurs in the continental United States because the amount for which the siblings sold their assets is given in dollars.
- No claims have been filed against the estate.
- The fourth child was ignored in the distribution of the assets, and the three children who sold their land received two-ninths of the land, which was worth $1,300 each. If the land had been distributed appropriately, then each child would have received one-sixth of the total assets. This would make each lot 12.5717 acres and worth $975.



- The jury does not decide who pays the compensation.
- Assets other than the 75.43 acres are negligible.
- The federal estate tax was paid before the assets were distributed.
- Inheritance tax is not taken into account because:

  a. The $1,300 that each of the other three children sold back to the mother was assumed to be the actual value of two-ninths of the land. The calculation of current value of the assets was directly based on this value.

  b. It is taken after compensation is awarded. Therefore, the jury does not need to take it into consideration.

Solutions 1 and 2:

- Farmland value increase was a representation of the increase in value of the inherited land since 1946 to 1949.
- The land was unaltered from its state in 1945.

Solution 3:

- The fourth child received one-sixth of the assets, 12.57166667 acres.
- The land would have been sold for $975 between 1946–1949.
- The $975 was held in a bank account accumulating interest.
- The interest on the account compounds yearly.
- The interest rate on the account changes every year, following the average values for interest.
- Interest for 2003 has already been accrued.

Solution 4:

- The estimated 2003 value was assumed to be correct due to past trends.

MODEL:

We decided that there are four possible ways to calculate the amount of compensation for the fourth son. Solution 1 calculates the current value of the land based on a ratio of the land values in the 1940s to that in 2003. Solution 2 calculates the present value of the acreage that the fourth child would have received. Solution 3 awards him the worth of one-sixth of the land in 1949 plus the interest accrued. Solution 4 gives him the worth of the land in 1949 plus inflation.

Solution 1:

First, the average value of an acre of land for the entire United States from 1946 to 1949 according to the Economic Research Service was calculated. Next, the average value of an acre of land for 2003 according to the Economic Research Service was divided by the average price per acre to obtain a conversion factor for the increased value of land. Then the value of the fourth child's assets was multiplied by the conversion factor to acquire a current value of the son's land, resulting in an equation:

$$V_{Total} = \left[\frac{P(y)}{\frac{P(y_1)+P(y_2)+P(y_3)+P(y_4)}{4}} \right] * F$$

where $P(y)$ is an average value of an acre of land for each year, y is the year, F is the fourth child's asset value, and V_{Total} is the total current value of the sons' land.

$$V_{Total} = \left[\frac{P(03)}{\frac{P(46)+P(47)+P(48)+P(49)}{4}} \right] * F$$

$$= \left[\frac{1270}{\frac{53+60+64+66}{4}} \right] * 975$$

$$= \left[\frac{1270}{60.75} \right] * 975$$

$$= 20382.72$$

Solution 2:

The average value per acre of land in 2003 according to the Economic Research Service was multiplied by the fourth son's share of the land, which was one-sixth of the total assets. The result is a current value of his land, which could be expressed as an equation: $V_{Total} = \mathrm{P}(y) * L$, where V_{Total} is the total current value of the son's land, $P(y)$ is an average value of an acre of land for the United States in 2003, and L is the fourth son's share of land in acres.

$$V_{Total} = \mathrm{P}(03) * L$$

$$= 1270 * 12.57166667$$

$$= 15966.02$$

Solution 3:

This solution is a calculation of the compensation according to the 1946 to 1949 value of land, which is $975, with the interest that would have accrued had the son received and sold his property at the same time as his three siblings. Since the interest was assumed added yearly, a slightly modified version of the equation below was used:

$$F = P(1 + i)^n$$

where F is the amount in the account after interest, P is the amount in the account at the first of the interest period, i is the interest rate, and n is the number of periods. The following is the modified version of the equation:

$$F_n = F_{n-1}(1 + i)$$

where F_n is the amount after interest and i is the interest rate. F_0, the initial amount in the account, is $975. A program was written to take the interest rates from 1946 to 2003 to determine the final amounts in an account that could have started in 1946, in 1947, in 1948, or in 1949. The program then averaged these four values, returning the value $13,972.10.

Solution 4:

This solution used the 1946 to 1949 consumer price index conversion factors to estimate his inheritance from the 1940s in 2003 dollars. The conversion values we found used 2003 as the base for the other conversion factors, meaning the conversion factor for 2003 is 1.000 and the other factors were based on this value. The 1946 to 1949 values were averaged to give 0.12175. We divided what his assets would have been in 1946 to 1949, $975, by this number to yield $8008.21 as his compensation.

DISCUSSION:

For the first two solutions, the use of farmland values was inaccurate due to the unknown location and type of the estate's land. Although Solution 1 uses a ratio of increase in land values, making it better for estimation than a straight calculation of present value as in Solution 2, it is unlikely that all types of land change at the same rate. Even in the information found for only farmland, regional changes from 1945 to 2003 varied widely. Solution 2 assumes that the fourth son would have kept the land and sold it for present value. Because the other three children sold their properties back to the mother, it is doubtful that the fourth son would have kept his share only to sell it after fifty-eight years. Once again, the land value varies regionally. If the son had been able to sell his land along with his other siblings, it is very likely that he would have put the money in a bank, as modeled by Solution 3. The interest rates used were average short-term yearly rates, but he may have put the money in an account that either compounds more or less often. The fourth scenario is improbable; the fourth child would have at least invested the money, instead of letting it depreciate.

Through this analysis, it was concluded that Solution 3 is the fairest compensation. Solution 3 emulates what the son probably would have done if he had received his inheritance at the proper time. $13,972.10 should be awarded to the son as his rightful inheritance.

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November 10, 2003

Dear Judge Robinson:

The jury has made a decision after much deliberation. This letter is intended to explain the jury's method for determining the appropriate amount bestowed upon the son of the deceased Noah Sentz. The sum that will be given to Mr. Sentz should be $13,972.10.

We came up with four different methods to determine the proper compensation. The first method calculates the value of the land using a designed ratio as a conversion factor of increase to assess the current value for the property, based on the prices given for the land in acres. The second method calculates the present value of the land that he would have received using current average value per acre of land in the United States. The third method awards the beneficiary his rightful share of the initial value of the land plus interest that would have been accrued. The fourth method gives the beneficiary his legal portion of the initial value of the estate taking inflation into account. Through the analysis of these processes, we concluded that the third method gives the beneficiary the fairest compensation. The third method emulates what the most probable actions the beneficiary would have taken had he received his inheritance at the proper time. A program was created to simulate the accumulation of interest with a changing rate over a 58-year period. The solution produced by the program showed the appropriate compensation that should be granted to the beneficiary.

Respectfully,

The Jury

SPREAD OF AN INFECTIOUS DISEASE

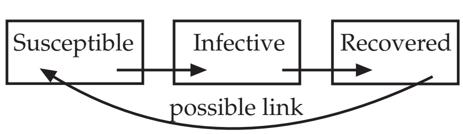
DAN TEAGUE & DOT DOYLE

Last spring it was SARS. This winter it's the flu. In the spring it's SARS again and the avian flu. The spread of AIDS has been in the news for the last 15 years. Every year students read articles in the news or see programs on television about the spread of some infectious disease. Since many modern curricula teach students to model with iterative or recursive equations, and modern calculators will generate sequences of values for these equations, modeling the spread of an infectious disease is an important and interesting project for students at both the Precalculus and Calculus levels.

Human history is crowded with the devastation of epidemics.

> In the 14th century, there were an estimate 25 million deaths in a population of 100 million Europeans attributed to an epidemic of bubonic plague. In 1520, the Aztecs suffered an epidemic of smallpox that resulted in the death of half their population of 3.5 million. When measles first came to the Fiji Islands in 1875 as a result of a trip to Australia by the King of Fiji and his son, it caused the death of 40,000 people in a population of 150,000. In the three-year period from 1918 to 1921, there were an estimated 25 million cases of typhus in the Soviet Union and about 1 in 10 victims died from the disease. In a world-wide epidemic of influenza in 1919, more than 20 million persons perished from the illness and subsequent attacks of pneumonia. (Olinick, *An Introduction to Mathematical Models in the Social Sciences*, 1978, page 349.)

The number of deaths from SARS last summer and from the flu in the fall pales in comparison to these historical epidemics, but the threat of new outbreaks is a part of our students' daily lives. This article will help them understand the basic mathematical models describing the spread of infectious diseases.



The Basic Model

The model for the spread of infectious diseases is known as a compartment model, since we think of people moving from one compartment to another. We assume we have a fixed population of N individuals through which an infectious disease is moving. Some of the people have the disease and are called Infectives. Some of the people do not yet have the disease but may catch it if they interact with an Infective. These are called Susceptibles. Some of the people may have already had the disease and have recovered from it. They are called Recovereds. For some diseases, the Recovereds develop immunity to the disease, while for others they return to the Susceptible group and can again come down with the disease. This model is illustrated in **Figure 1**.

The Loss-Gain Equation

Compartment models can be modeled nicely by the difference (iterative) equations with the form *New Value* = *Old Value* + *Gain* – *Loss* or $Y_{n+1} = Y_n + Gain - Loss$. Most of the iterative models studied in Precalculus courses have this form. A couple of typical Precalculus examples not related to the spread of disease illustrate this structure.

FIGURE 1. A COMPARTMENT MODEL

EXAMPLE 1: Jo-Ann strained her knee playing tennis and her doctor has prescribed ibuprofen to reduce the inflammation and control pain. Jo-Ann is instructed to take one 220-milligram ibuprofen tablet every 4 hours for 10 days. As the drug circulates, it has its anti-inflammatory effects on Jo-Ann's knee, and as its passes through the kidneys it is filtered out of Jo-Ann's system. During any given time period, the kidneys filter the impurities (in this case, the kidneys consider the drug an impurity) from a fixed amount of blood. If Jo-Ann's kidneys filter 65% of the drug in her body every 4 hours, how much of the drug will be in Jo-Ann's system after 96 hours?

In this example, we have

New Value = Old Value + Gain – Loss

$$Y_{n+1} = Y_n + 220 - 0.65Y_n$$

We also need an initial condition, $Y_0 = 220$. The difference equation above can be simplified to $Y_{n+1} = 0.45Y_n + 220$.

EXAMPLE 2: Suppose you are interested in purchasing a car and need a $5000 loan. The lending agency is going to charge you interest each month and you are going to make a payment each month. You plan to pay $100 each month until the loan is paid off. Suppose the interest rate is 0.75% per month. How long will it take you to repay the loan?

The second example has the Loss-Gain equation

New Value = Old Value + Gain – Loss

$$Y_{n+1} = Y_n + 0.0075Y_n - 100$$

with the initial condition, $Y_0 = 5000$. The difference equation above can be simplified to $Y_{n+1} = 1.0075Y_n - 100$.

First Model for the Spread of Infectious Disease

Our first model will be a simplified compartment model with only two compartments, *Susceptible* (S) and *Infective* (I). This model is illustrated in **Figure 2**. We have a population of size N, so $S + I = N$. This model could represent an animal population in which the infected animal does not leave the herd, or the case of a mild cold spreading through a college dormitory.

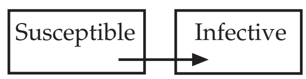


FIGURE 2. SIMPLE MODEL WITH TWO STATES

We can model the number of *Susceptibles* after n intervals of time with $S_{n+1} = S_n + Gain - Loss$ and the number of *Infectives* with $I_{n+1} = I_n + Gain - Loss$. In our simple 2-compartment model, there is no *Gain* for the *Susceptibles* since nothing enters that compartment and there is no *Loss* for the *Infectives* since nothing leaves that compartment, so our two equations are $S_{n+1} = S_n - Loss$ and $I_{n+1} = I_n + Gain$. What our diagram also makes obvious is that the *Gain* for *Infectives* is the same as the *Loss* for *Susceptibles*, since the new *Infectives* are coming from the *Susceptible* compartment. How does a *Susceptible* become an *Infective*?

Modeling the Transition Rate

For most infectious diseases, transmission happens when an *Infective* comes in "contact" with a *Susceptible*. This contact could be physical contact as in many STDs, or contact via a cough or door handle, or bites from the same mosquito. Not all *Infectives* interact with all *Susceptibles*, but the larger the sub-population of *Susceptibles*, the greater the probability of an interaction. Likewise, the larger the sub-population of *Infectives*, the greater the probability of an interaction. Since not all contacts result in transmission of the disease, we can describe the rate of transmission as $\alpha \cdot S \cdot I$, where the value of α carries with it both the probability of interaction between the two and the probability of transmission given an interaction.

Our Loss-Gain equations then become $S_{n+1} = S_n - \alpha \cdot S_n \cdot I_n$ and $I_{n+1} = I_n + \alpha \cdot S_n \cdot I_n$. If we focus on the number of *Infectives*, we can rewrite the equation in terms of I only as,

$$I_{n+1} = I_n + \alpha \cdot (\mathrm{N} - I_n) \cdot I_n.$$

We can use this model to investigate this simple, two compartment model. Then we will modify the model to include *Recovereds*.

Investigating the Two Compartment Model

In the Precalculus class, we need specific values for N and α to iterate the equation above. Let's consider $N = 1000$ and $\alpha = 0.001$ with n representing days. We will start with only one *Infective*. So $I_{n+1} = I_n + 0.001 \cdot (1000 - I_n) \cdot I_n$ with $I_0 = 1$. If we look at the graph of the iteration **(Figure 3),** we see the progression of the disease through the population of 1000.

Notice that all of the individuals in the population eventually get sick. The shape of the curve is a classic example of logistic growth. This is an important model for students to be aware of. If we look carefully at the graph or look at differences in a table of values **(Table 1)**, we can see when the epidemic is growing most rapidly. This is when the epidemic is most obvious and causes the most concern (or panic) in the population.

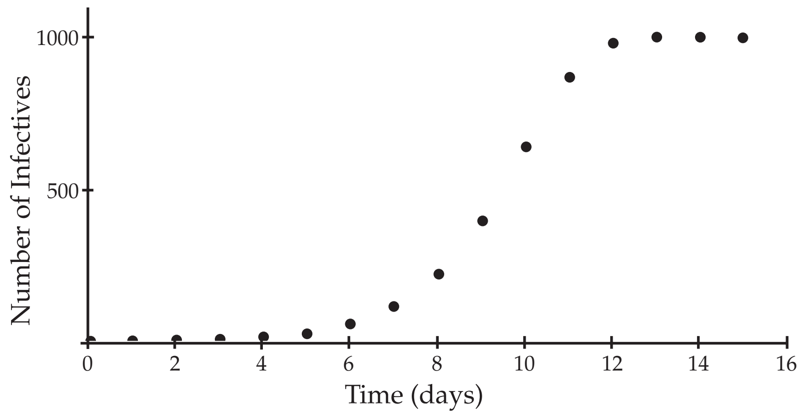


FIGURE 3. THE SPREAD OF THE DISEASE

n days	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
I_n	1	2	4	8	16	31	62	120	226	401	641	871	983	1000	1000
ΔI_n	~	1	2	4	8	15	31	58	106	175	240	230	112	17	0

TABLE 1. NUMBER OF INFECTIVES (I_N ROUNDED TO INTEGER) AND CHANGE IN INFECTIVES (ΔI_N)

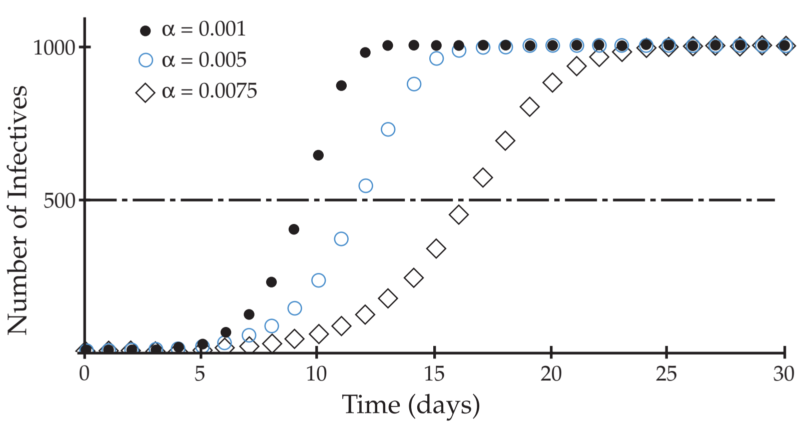


FIGURE 4. COMPARING TRANSMISSION RATES $\alpha = 0.001$, $\alpha = 0.005$, AND $\alpha = 0.0075$

The greatest growth in this example occurs when approximately half of the population has the disease. Is this chance or does it happen all the time? If we change the growth rate to $\alpha = 0.005$ and $\alpha = 0.0075$, we can see what happens **(Figure 4)**.

We see that in all three cases, the maximum growth (largest difference between successive values) happens around $I = 500$. Why?

Our Loss-Gain equation gives the answer. If, $I_{n+1} = I_n + 0.001 \cdot (1000 - I_n) \cdot I_n$, then in each time interval we add a value proportional to $S \cdot I = (N - I) \cdot I$. The function $f(I) = (N - I) \cdot I$ has its largest value at $I = \frac{N}{2}$.

A More Realistic Model with Recovery

A more realistic model, known as the *SIR* model, included *Recovereds*. The compartment model has three compartments with transitions between adjacent compartments **(Figure 5)**.

FIGURE 5. THREE COMPARTMENT (*SIR*) MODEL

Our Loss-Gain Equations can be developed as before. There is no Gain for *Susceptibles*, no Loss for *Recovereds*, and the Gain for *Infectives* is the Loss for *Susceptibles* while the Loss for *Infectives* is the Gain for *Recovereds*. The only real change is to add a rate of recovery, β. During each time interval, a proportion of those infected will recover. They can no longer spread the disease and are immune to catching it again. For many diseases, if the infected have the disease for k days, the recovery rate is estimated as $\beta = \frac{1}{k}$. Unfortunately, for some diseases, "recovery" may mean death due to the disease. Our Loss-Gain equations become:

$$S_{n+1} = S_n + Gain - Loss$$
$$I_{n+1} = I_n + Gain - Loss$$
$$R_{n+1} = R_n + Gain - Loss$$

so

$$S_{n+1} = S_n - Loss$$
$$I_{n+1} = I_n + Gain - Loss$$
$$R_{n+1} = R_n + Gain$$

and

$$S_{n+1} = S_n - \alpha \cdot S_n \cdot I_n$$
$$I_{n+1} = I_n + \alpha \cdot S_n \cdot I_n - \beta \cdot I_n$$
$$R_{n+1} = R_n + \beta \cdot I_n$$

If $\alpha = 0.001$ and $\beta = 0.2$ (it takes 5 days to recover), what is the progression of the disease? (See **Figure 6**.)

There is almost nothing we can do to reduce the value of β, so to affect the

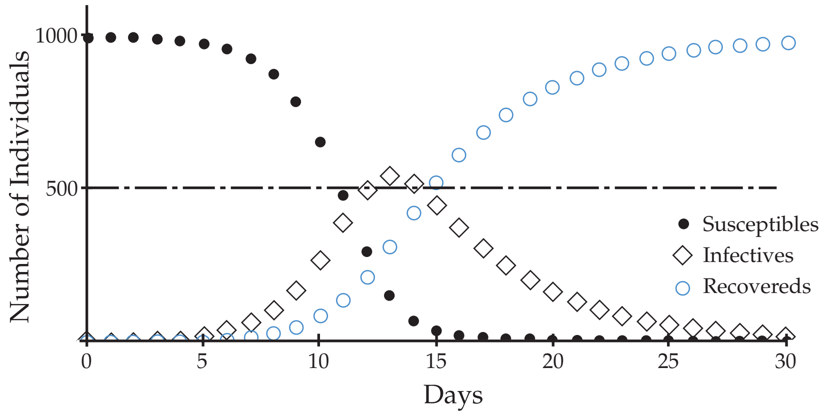


FIGURE 6. NUMBER OF *SUSCEPTIBLES*, *INFECTIVES*, AND *RECOVEREDS* WITH $\alpha = 0.001$ AND $\beta = 0.2$

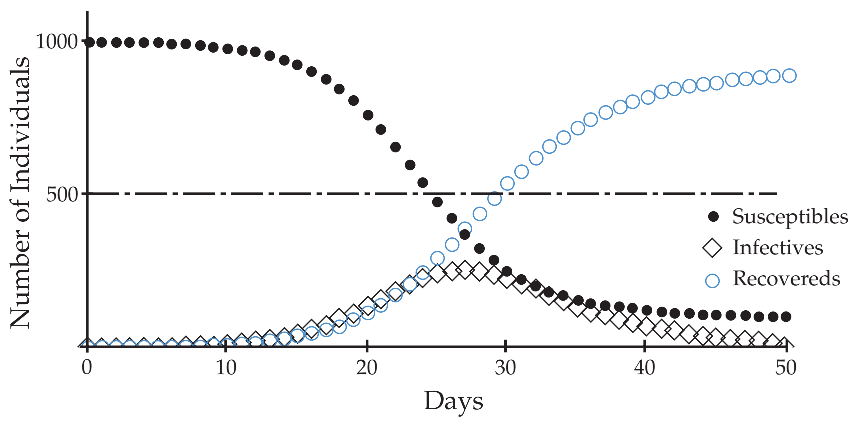


FIGURE 7. $\alpha = 0.0005$

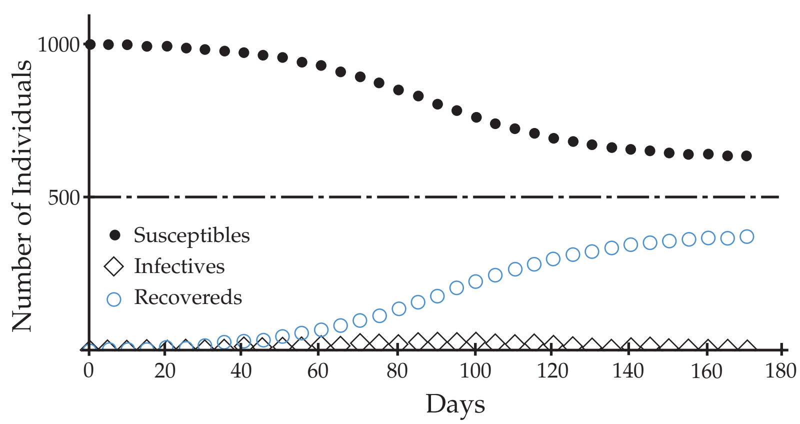


FIGURE 8. $\alpha = 0.00025$

spread of the disease, we have to reduce the value of α. Suppose we cut the value of α in half ($\alpha = 0.0005$) by wearing masks to reduce the spread of air-borne particles. How does this affect the number of people who ultimately get the disease? (See **Figure 7.**)

Notice that the disease took much longer to move through the population and not everyone seems to get the disease. After 50 days there are still around 100 people who did not get the disease. If we reduce the transmission rate further ($\alpha = 0.00025$) by isolating those infected, will even fewer catch the disease? (See **Figure 8.**)

Now, we see that fewer than half of the population get the disease, even after 5 months. Is there some critical value for α that will either create an epidemic in which everyone gets sick or have only a portion of the population become ill?

Analyzing the Equations

If we look at the defining Loss-Gain equations, we may be able to see beyond the graphs.

$$S_{n+1} = S_n - \alpha \cdot S_n \cdot I_n$$

$$I_{n+1} = I_n + \alpha \cdot S_n \cdot I_n - \beta \cdot I_n$$

$$R_{n+1} = R_n + \beta \cdot I_n$$

The number *Infectives* will increase if $\alpha \cdot S_n \cdot I_n - \beta \cdot I_n$ is greater than zero and will decrease if $\alpha \cdot S_n \cdot I_n - \beta \cdot I_n$ is less than zero. This term seems to be the determining factor. If $\alpha \cdot S_n \cdot I_n - \beta \cdot I_n > 0$, then $(\alpha \cdot S_n - \beta)I_n > 0$. Since S_n and I_n are both always non-negative, $(\alpha \cdot S_n - \beta)I_n > 0$ means that $(\alpha \cdot S_n - \beta) > 0$, or $S_n > \frac{\beta}{\alpha}$. This is **the** key to an epidemic! Look back at the graphs. The number of *Infectives* begins to decrease whenever the number of available *Susceptibles* drops below $\frac{\beta}{\alpha}$. In Figure 6, $\frac{\beta}{\alpha} = 200$, so the disease doesn't begin to slow down until after 80% of the population has the disease. In Figure 7, $\frac{\beta}{\alpha} = 400$, so the disease begins to slow down when 60% remain unaffected, while in Figure 8, $\frac{\beta}{\alpha} = 800$, and the disease begins to decline after only 20% are affected and 80% remain well.

A Calculus Model

The differential equations modeling this system are similar to the difference equations used before. The iterative equations

$S_{n+1} = S_n - \alpha \cdot S_n \cdot I_n$
$I_{n+1} = I_n + \alpha \cdot S_n \cdot I_n - \beta I_n$
$R_{n+1} = R_n + \beta I_n$

become the differential equations

$$\frac{dS}{dt} = -\alpha \cdot S \cdot I$$

$$\frac{dI}{dt} = \alpha S \cdot I - \beta I \text{ with } S(t) + I(t) + R(t) = N$$

$$\frac{dR}{dt} = \beta \cdot I$$

Calculus students should use Euler's method (with $\Delta t = 1$) to investigate the effects of altering the values of α and β on the spread of the disease. Their investigation should lead them to some conjectures about the progress of the disease that can be confirmed or rejected after solving the differential equations.

As before, the equation modeling the change in *Infectives* is the key to understanding the situation. From the differential equation $\frac{dI}{dt} = \alpha SI - \beta I$, we know that $\frac{dI}{dt} = I(\alpha S - \beta) > 0$ and the number of *Infectives* is increasing whenever $S > \frac{\beta}{\alpha}$. As before, this is **the** important fact of the growth of the epidemic. The key to changing the dynamics of the disease, either by changing S_0, α or β was to get $S < \frac{\beta}{\alpha}$. If you look back at the graphs and numerical values, you will see that $S = \frac{\beta}{\alpha}$ is a critical value.

The advantage of the differential equations model over the iterative model (either the Precalculus version or Euler's method) is in finding closed form models by solving the equations. To find the function determining the number of *Infectives*, we can eliminate t from the equations by solving for $\frac{dI}{dS}$. We know that

$$\frac{dI}{dS} = \frac{\left(\frac{dI}{dt}\right)}{\left(\frac{dS}{dt}\right)} = \frac{\alpha SI - \beta I}{-\alpha SI} = -1 + \frac{\beta}{\alpha S}.$$

Solving for $I(S)$, we have

$\int dI = \int -1 + \frac{\beta}{\alpha S} dS$, which simplifies to

$I(S) = -S + \frac{\beta}{\alpha}\ln(S) + C.$

If $t = 0$, we have S_0 and I_0 *Susceptibles* and *Infectives*, respectively, and we know that

$$I(S) = S_0 + I_0 - S + \frac{\beta}{\alpha}\ln\left(\frac{S}{S_0}\right).$$

What does this function look like? We already know that $I'(S) = -1 + \frac{\beta}{\alpha}\frac{1}{S}$.

Since $I''(S) = -\frac{\beta}{\alpha}\frac{1}{S^2}$ is always negative, we know that the function $I(S)$ is always concave down and has its maximum value at $S = \frac{\beta}{\alpha}$. If $I' > 0$ the number of *Infectives* increases and if $I' < 0$, the number of *Infectives* is decreasing.

The graph of $(S)I = S_0 + I_0 - S + \frac{\beta}{\alpha}\ln\left(\frac{S}{S_0}\right)$ can be deceiving since S is always decreasing.

So, when $S < \frac{\beta}{\alpha}$, the epidemic will begin to wind down. As long as $S > \frac{\beta}{\alpha}$, the epidemic will continue to build. The value $\frac{\beta}{\alpha}$ is the ratio of the rate at which *Infectives* become recovered and *Susceptibles* become infected. Notice that if $S = \frac{\beta}{\alpha}$, then both $\frac{dI}{dS}$ and $\frac{dI}{dt}$ are equal to zero. So, again, we see that the ratio of β to α is important in the spread of the disease. The best way to restrict the spread of the epidemic is to affect this ratio.

If there is a large enough population of *Susceptibles*, $S > \frac{\beta}{\alpha}$, then the number of infected individuals will increase. There must be a sufficient number of *Susceptibles* available for the epidemic to develop. This is why separating the infected from *Susceptibles* by quarantine is important in halting the disease. Notice that the initial number of *Infectives* does not seem to matter, since it does not appear in the derivative. The epidemic will end, naturally (that is, without intervention), when the number of available *Susceptibles* is too small. This does not mean that everyone will eventually get the disease. Once $S = \frac{\beta}{\alpha}$, the epidemic will begin to wind down. By isolating *Infectives*, we are effectively reducing the number of available *Susceptibles*.

Maximum Proportion Ill at One Time

What proportion of the population will have the disease when it is at its peak? This proportion affects the public's perception of the seriousness of the outbreak. We can use our equation $I(S)$ and evaluate it at $S = \frac{\beta}{\alpha}$. So,

$$I\left(\frac{\beta}{\alpha}\right) = S_0 + I_0 - \frac{\beta}{\alpha} + \frac{\beta}{\alpha}\ln\left(\frac{\beta/\alpha}{S_0}\right).$$

In our example above, we had $\frac{\beta}{\alpha} = \frac{0.2}{0.001} = 200$, $S_0 = 999$, and $I_0 = 1$, so the maximum number infected will be

$$I(200) = 1000 - 200 + 200\ln\left(\frac{200}{999}\right) \approx 478$$

or almost half of the population sick at one time. If, however, we have a situation in which $\frac{\beta}{\alpha} = 500$ (by wearing masks, for example), then we would have

$$I(500) = 1000 - 500 + 500\ln\left(\frac{500}{999}\right) \approx 154$$

or 15% of the population. This is still a lot, but the size of the epidemic is dramatically reduced and the sense of panic that may occur when many people are ill at once is markedly reduced.

Total Number Falling Ill

The total number infected can be found using our function $I(S) = S_0 + I_0 - S + \frac{\beta}{\alpha}\ln\left(\frac{S}{S_0}\right)$. The epidemic is over when $I = 0$. So, for each situation, we can use numerical methods to solve

$$S_0 + I_0 - S + \frac{\beta}{\alpha}\ln\left(\frac{S}{S_0}\right) = 0.$$

In our initial example, $\frac{\beta}{\alpha} = 200$, then

$1000 - S + 200\ln\left(\frac{S}{999}\right) = 0$ at $S \approx 7$, so essentially everyone will eventually contract the disease.

If we increase $\frac{\beta}{\alpha}$ to 400 by washing hands and other practices of good hygiene we have

$$1000 - S + 400 \cdot \ln\left(\frac{S}{999}\right) = 0$$

at around $S = 107$. This means 107 individuals do not become infected. And if we can raise $\frac{\beta}{\alpha}$ to 800 (Figure 8) we have

$$1000 - S + 800 \cdot \ln\left(\frac{S}{999}\right) = 0$$

at around $S = 626$. Fewer than half of the *Susceptible* population become ill.

We conclude this edition of *Everybody's Problems* with a handout for Precalculus students suggesting activities that will lead them to an understanding of the material developed in this article. A follow-up handout for Calculus students assumes they have answered the questions posed to the Precalculus group using Euler's Method. ❏

Everybody's Problems concerns teaching high school mathematics courses with real-world problems, particularly problems that are suitable for students at all levels.

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Project Handout for Precalculus Students

By considering the mechanism of transfer, together we have developed a system of Loss-Gain iterative equations

$S_{n+1} = S_n - \alpha \cdot S_n \cdot I_n$

$I_{n+1} = I_n + \alpha \cdot S_n \cdot I_n - \beta I_n$

$R_{n+1} = R_n + \beta I_n$

that model the spread of an infectious disease. For our investigation, use $N = 1000$, $S_0 = 9999$, $I_0 = 1$, $R_0 = 0$, $\alpha = 0.001$ and $\beta = 0.2$. We will alter these values to see what effect they have on the spread of the disease.

Computer (or Calculator) Investigation

1. Iterate the defining equations above to get baseline information about the behavior of the three groups. How does the progress of the disease change if $N = 800$, $N = 300$, or $N = 100$. For each of these populations, estimate the total number that eventually come down with the infectious disease and the maximum proportion of the population ill with the disease at one time.

2. With $N = 1000$, suppose the disease is more difficult to catch than our model suggests. Use $\alpha = 0.0005$. How does this affect the total number that come down with the disease and the proportion ill with the disease at one time? Suppose the disease is easy to catch, with $\alpha = 0.005$, how does this alter the progress of the disease? How low must α be before fewer than half of the population actually gets ill?

3. Using $N = 1000$ and $\alpha = 0.001$ again, suppose a treatment is found that cuts the recovery time in half. How does this affect the total number that come down with the disease and the maximum proportion ill at one time?

4. For the most part, we can't really affect the value of β very much. Recovery time is often little affected by our ministrations. The transfer rate can be affected in a number of ways. Many people in areas where an infectious disease is widespread try to reduce the spread of the disease by wearing masks. The Canadian Broadcast Corporation (CBC) did a report on the ability of masks to reduce the number of microscopic particles breathed in. They found that the standard dust mask reduced the number of airborne particles by 13 percent, a dentist's mask by 32 percent, a surgical mask by 62 percent, and an N-95 mask (like those used by the Army in Iraq) by 98 percent. Could wearing masks significantly alter the spread of the disease or must a vaccine be found?

Additional Questions for Calculus

In our computer or calculator investigation using Euler's Method (iterations), we have developed a number of conjectures about what would happened to the progress of the disease under different conditions by changing the values of N, α and β. We want to verify those results and learn some crucial facts about epidemics by using Calculus.

1. When is $I(t)$ increasing and when is it decreasing? What does this mean about the spread of the disease? Using this result, explain why isolation and quarantine will be effective against SARS.

2. We have equations for $\frac{dI}{dt}$ and $\frac{dS}{dt}$. Use them to find $\frac{dI}{dS}$. Solve this differential equation for $I(S)$. Find $I'(S)$ and $I''(S)$ and use these derivatives to determine when the number of infected will begin to decrease. Does this match your solution from question 1? Explain any differences you see.

3. Use the function $I(S)$ to determine the maximum number of individuals ill at one time. What proportion of the population will be sick simultaneously? Are these results consistent with your graphs in the computer investigation?

4. How many *Infectives* will there be when the epidemic is over? Use this idea to determine the proportion of the population infected under the different situations considered in the computer investigation. How effective do the masks need to be to seriously affect the progress of the disease? Is it realistic to think isolation and wearing masks can work or must a vaccine be found?

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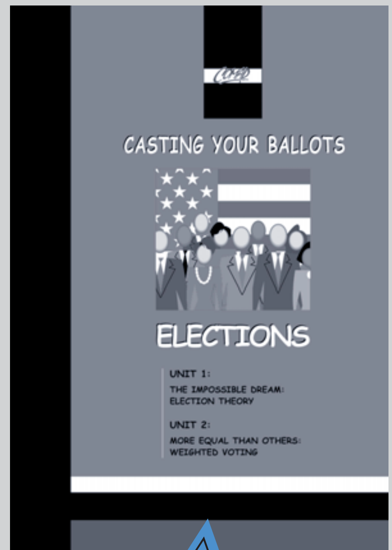

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